TIME OPTIMAL CONTROL

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In this talk it is my purpose to acquaint you with some of the principle results in the theory of time optimal control. It is not my intention to go into details. A brief history of the problem, references, the precise statement of the theorems, and their proofs can be found in [1]. Important contributions to this problem have been made by Bushaw, Bellman, Krasovski, Gamkrelidze, Pontrjagin, and Boltyanskii.

Let me begin with a fairly general statement of the problem. The control system is described by a system of differential equations

$$\dot{x}(t) = X(x(t), u(t), t) \tag{1}$$

where $x(t)$ is an n-vector and $u(t)$ is an r-vector. The function u is called the steering function and is to be selected from a class U of functions in such a manner as to achieve some objective. The system starts initially in the state x^o. Let $x(t) = x(t, x^o, u)$ denote the solution of (1) satisfying $x(0) = x^o$. The objective which is to be achieved at some later time may be described by an equation

$$g(x(T), T) = 0. \tag{2}$$

The time T at which this objective is achieved will depend upon the initial state x^o and the steering function u. For a given initial state x^o a steering function u will be said to be *optimal* if it minimizes a functional

$$F(u, x^o) = \int_0^T h(x(t), u(t), t)\, dt$$

This is an implicit problem in the calculus of variations. By this we mean that the functional to be minimized is described implicitly. In the time optimal problem $h = 1$ and the functional to be minimized is the time T to achieve the objective.

In a trajectory problem one would be given x^o and would want to know optimal steering u as a function of time. This is the problem in a mathematical study that one attacks first.

In a feedback or automatic control system the steering function is to depend upon the state of the system. We confine ourselves to the completely deterministic problem and assume that the system can at all times measure its state $x(t)$

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and the steering is then $u(t) = Q(x(t))$. The differential equation of the system is

$$\dot{x} = X(x, Q(x), t) \tag{3}$$

The automatic steering $Q(x)$ is optimal—possibly only for a restricted region of states x—if $u(t) = Q(x(t))$ is optimal in the sense described above. To be more precise, let $x(t, x^o, Q)$ be the solution of (3) satisfying $x(0) = x^o$. Then $Q(x)$ is optimal if $u(t) = Q(x(t, x^o, Q)$ is optimal for all x^o. Again this may be for only a restricted region of initial states. The problem of determining optimal feedback or automatic steering has been called by mathematicians the "synthesis problem".

As we might expect not too much is known about the general problem—either the trajectory or the feedback problem—except in what is essentially the linear case. In this case the differential equation of the system is of the form

$$\dot{x} = A(t)x + B(t)u + f(t) \tag{4}$$

where x and $f(t)$ are n-vectors, u is an r-vector, $A(t)$ is an $n \times n$-matrix and $B(t)$ is an $n \times r$-matrix. Let the class U of allowable steering functions be measurable functions whose components each satisfy $|u_i| \leq 1$, $i = 1, \cdots, r$. Assume also that $A(t)$, $B(t)$, and $f(t)$ are continuous for all $t \geq 0$. A moving particle $z(t)$ in the phase space is given. Assume also that $z(t)$ is continuous for all $t \geq 0$. Initially $x(0) = x^o$ and the objective is to have $x(T) = z(T)$, $T > 0$, and to do this in minimum time. This, for instance, is the problem of landing on a satellite. Then T is given implicitly by the equation

$$z(T) = X(T)x^o + X(T)\int_0^T X^{-1}(t)B(t)u(t)\,dt + X(T)\int_0^T X^{-1}(t)f(t)\,dt,$$

where $X(t)$ is the principal matrix solution of $\dot{x} = A(t)x$. Defining

$$w(T) = X^{-1}(T)z(t) - x^0 - \int_0^T X^{-1}(t)f(t)\,dt,$$

and

$$Y(t) = X^{-1}(t)B(t),$$

we have as the equation for T

$$w(T) = \int_0^T Y(t)u(t)\,dt.$$

T is the functional to be minimized. Thus we are lead to a study of the set $\Gamma(T)$ of all $\int_0^T Y(t)u(t)\,dt$ for $u \in U$. Let U^o be the set of all $u \in U$ with $|u_j| = 1$, $j = 1, \cdots, r$. U^o is the set of so-called "bang-bang" steering functions. Let $\Gamma^o(T)$ be the range of $\int_0^T Y(t)u^o(t)\,dt$ with $u^o \in U^o$. The set $\Gamma^o(T)$ tells us what can be accomplished by bang-bang steering in time T.

It can then be shown that $\Gamma^o(T)$ is a compact, convex set, and from that it can be shown that

$$\Gamma^o(T) = \Gamma(T).$$

This means that *anything that can be accomplished in time T by the allowable steering can be accomplished in the same time T by bang-bang steering.*

If for some $T \geq 0$, $w(T) \in \Gamma(T)$ it then follows without difficulty that there is a minimal $T^* \geq 0$ with $w(T^*) \in \Gamma(T^*)$ and an optimal steering function u^*. It can be shown also that $w(T^*)$ must be on the boundary of $\Gamma(T^*)$, and hence that every optimal steering function is of the form

$$U^*(t) = \operatorname{sgn}[\eta Y(t)], \tag{5}$$

where η is some nonzero n-vector. The signum is to be taken by components. Thus, the jth component $u_j(t)$ of $u(t)$ is 1 when the jth component of $\eta Y(t)$ is positive, -1 when it is negative, and undetermined by (5) when $\eta Y(t) \equiv 0$ on an interval of positive length.

Let us look at the converse question. Suppose $u(t) = \operatorname{sgn}[\eta Y(t)]$, $\eta \neq 0$, $w(T) = \int_0^T Y(t)u(t)\,dt$, and $w(t_1) \neq \int_0^{t_1} Y(t)u(t)\,dt$ for all $0 \leq t_1 < T$. With this particular steering function the objective is reached in finite time. Is it then true that $u(t)$ is optimal? In general, the answer is no. But it is easy to see that with the following additional assumptions

(i) $\eta Y(t) \not\equiv 0$ on any interval of positive length within $[0, T]$,

(ii) $\eta[w(t) - w(T)] \geq 0$ for $t < T$,

then a $u(t)$ of this form is optimal. Suppose, for instance, that $w(t) \equiv -x^o$, which is the case if the objective is to reach the origin and $f(t) \equiv 0$. Then condition (ii) is satisfied, and this leads naturally to the consideration of systems with the property that:

$$\eta Y(t) \equiv 0 \text{ on an interval of positive length implies } \eta = 0.$$

Such systems have been called *proper* systems. In the case of constant coefficients

$$\dot{x} = Ax + Bu \tag{6}$$

where A and B are constant matrices, a system is proper if and only if

$$\begin{matrix} b^1, Ab^1, \cdots, A^{n-1}b^1 \\ \vdots \qquad\qquad \vdots \\ b^r, Ab^r, \cdots, A^{n-1}b^r \end{matrix} \tag{7}$$

contains a set of n linearly independent vectors. The vectors $b^1, \cdots, b^r$ are the column vectors in B.

Proper systems (4) have a number of noteworthy control properties. One of these is that *every proper system is completely controllable.* Complete controllability is the following property: Consider first that all constraints on the steering u have been removed. Let t_0, x^o and t_1, x^1 be given, $t_1 > t_0 \geq 0$. If $x(t_0) = x^o$, then there is a steering function u such that $x(t_1) = x^1$. The system can be moved from any one state to any other as rapidly as one may wish, if the system is a proper system.

Let us assume that $w(t)$, $t \geq 0$, is bounded. One example of importance is when $w(t) \equiv -x^o$. It can then be shown that, if $\dot{x} = A(t)x$ is stable (all solutions bounded for $t \geq 0$), then for each initial condition x^o there is always a $u \in U$ that guides the system to hit the moving particle $z(t)$; that is, there is always a $T > 0$ with the property that $w(T) \in \Gamma(T)$.

A property that is stronger than that of being proper is the assumption that no component of $\eta Y(t)$, $\eta \neq 0$, is identically zero on an interval of positive length. Such systems are called *normal* systems. In this case, optimal steering is uniquely determined by (5). For each x^o there is at most one optimal steering function x. If A and B are constant matrices, then the control system is normal if and only if each row of vectors in the array (7) is linearly independent.

Assume now that (6) is normal and the objective is to reach the origin in minimum time. If, in addition, $\dot{x} = Ax$ is stable, then we know for each initial state x^o there is unique, optimal steering. Thus, in principle, the synthesis problem can be solved by starting at the origin, using steering of the form (5) and running the system backwards (replace t by $-t$). Each point x can be reached in this manner and unique optimal feedback steering $Q(x)$ is assigned to each state x.

Beyond the linear case discussed above not too much is known. Perhaps the most significant general result is *Pontryagin's maximum principle* (see [2]).

We shall describe this principle for the special case of the time optimal problem. The principle itself is an extension of Weierstrass's criterion in the calculus of variations and is an interesting example of Lagrange multipliers. Let the control system be described by the differential equation

$$\dot{x} = X(x, u), \tag{8}$$

where the allowable steering functions u are piecewise continuous with values in a closed region Ω of r-space. Introduce the Hamiltonian

$$H(\psi, x, u) = \psi \cdot X(x, u)$$

where ψ is an n-vector. Define

$$M(\psi, x) = \operatorname*{Max}_{u \in \Omega} H(\psi, x, u).$$

Consider now the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial \psi}, \qquad \dot{\psi} = \frac{\partial H}{\partial x}. \tag{9}$$

Thus we have adjoined the equation for ψ to (8) to obtain this Hamiltonian system. Assume that $x(t)$ is a solution of (8) corresponding to a particular choice of allowable steering. Let $x(t_0) = x^o$ and $x(t_1) = x^1$. If this steering function is optimal in the sense that it minimizes the transit time from x^o to x^1, then Pontryagin's maximum principle states that there is a solution $\psi(t)$ of (9) with the property that

$$H(\psi(t_0), x(t_0), u(t_0)) \geq 0$$
$$H(\psi(t), x(t), u(t)) \equiv M(\psi(t), x(t)).$$

It turns out that $H(\psi(t), x(t), u(t))$ is a constant. For the system (6) and Ω a convex, compact polyhedron and the condition that Bw, ABw, $\cdots$, $A^{n-1}Bw$ be linearly independent when w is any vector in the direction of a side of Ω, Pontryagin has shown that optimal steering is unique and piecewise constant with values which are the vertices of the polyhedron Ω.

As yet there do not appear to be any significant results on structural properties for optimal control of nonlinear systems.

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