REACHABLE ZONES IN AUTONOMOUS DIFFERENTIAL SYSTEMS

By Emilio O. Roxin

I shall speak about a work done in collaboration with Mrs. Vera W. de Spinadel, concerning what we called "theory of reachable zones in autonomous differential systems".

Let us consider the first order differential equation

$$\dot{x} = \frac{dx}{dt} = f(x) + w(x, t) \tag{1}$$

where x, $f(x)$, $w(x, t)$ are n-dimensional real vectors and t a real variable. We may think of this system as describing a physical system, which in its free evolution satisfies the equation

$$\dot{x} = f(x) \tag{2}$$

which will be called *fundamental system*. The term $w(x, t)$ may be a perturbation or control term; accordingly we shall consider the *perturbator system*

$$\dot{x} = w(x, t). \tag{3}$$

Equation (1) will be called *complete system* or *perturbated system* and will be considered as obtained by *superposition* of (2) and (3).

Each of Equations (1), (2), and (3) define a vector field and we shall speak about the *fundamental vector* (2) and the *perturbator vector* (3). We shall suppose that these equations are defined in a certain region of the n-space and have there unique solutions, which depend continuously on the initial conditions and parameters that may appear in the equation. The solution-curves of equation (2) will be called *fundamental curves* and the solutions of (1) will be simply called *paths*.

The roles of $f(x)$ and $w(x, t)$ will be different in (1): for a certain problem, $f(x)$ will be a fixed function, meanwhile $w(x, t)$ will vary arbitrarily, restricted by certain conditions inherent to the problem considered. For a given initial point x_0 we have, consequently, an infinity of paths (solutions of (1)), starting at x_0, according to all possible choices of $w(x, t)$.

Note that, when we consider the whole set of paths starting at x_0, we may exclude the explicit dependence of $w(x, t)$ on x, since for each solution x there is a given function $x(t)$, and consequently $w(x, t) = w(x(t), t) = w^*(t)$. In what follows, we shall simply use $w(t)$, except in those cases where for some special reason it could be convenient to state explicitly the dependence on x.

If we do not restrain the function $w(t)$, nothing interesting is to be expected, as indicated by the following:

Theorem 1: *Let (1) be a system of ordinary differential equations, where $f(x)$ is a fixed function, and let*

$$x = \xi(t) \qquad (0 \leq t \leq t_1)$$

be the equation of a (sufficiently smooth) curve. Then, there is a function $w(t)$, defined for $0 \leq t \leq t_1$, such that the system (1), with $x(0) = \xi(0)$, has as solution precisely the curve $x = \xi(t)$.

The proof is immediate, choosing

$$w(t) = \dot{\xi}(t) - f[\xi(t)]$$

The conclusion is that every curve of the n-space is a possible solution of (1). To make the problem interesting, we must restrain in some way the function $w(t)$. In all these considerations, the curve to be concerned about has to fulfill the conditions necessary for being a solution of a differental equation system. We shall not repeat these conditions every time.

Restrictive conditions

We shall consider the general type of conditions for $w(t)$ vector of components $w_i(t)$

$$(4) \qquad \begin{aligned} F_i(w_1, w_2, \cdots, w_n) &= 0 \qquad (i = 1, 2, \cdots, p) \\ F_i(w_1, w_2, \cdots, w_n) &> 0 \qquad (i = p+1, \cdots, q) \end{aligned}$$

or, in parametric form

$$(5) \qquad w_i(t) = \Gamma_i(x_1, \cdots, x_n; \alpha_1, \cdots, \alpha_s)$$

the α_i being arbitrary functions of t, subjected eventually to inequality conditions of the type

$$\alpha_i < k_{i1}, \qquad \alpha_i > k_{i2}.$$

Reachable points

DEFINITION 1: Given the system (1), that is, the function $f(x)$, and certain restrictive conditions for $w(x, t)$ of the form (4) or (5), the point x_1 will be called *reachable from* x_0, if there is a function $w_1(t)$ which satisfies the given conditions, and a value $t_1 > 0$ such that the system

$$(6) \qquad \dot{x} = f(x) + w_1(t) \qquad x(0) = x_0$$

has a solution $x(t)$ such that $x(t_1) = x_1$.

In other words, x_1 is reachable from x_0 if it is on the positive half-path of (6).

DEFINITION 2: With the same conditions as before, the set of points x reachable from x_0 will be called *reachable zone from* x_0.

DEFINITION 3: With the same conditions, the set of points Z which has the following property: if i) $x_0 \in Z$ and ii) x_1 is reachable from x_0, then $x_1 \in Z$, will be called *generalized reachable zone* (g.r.z.)

Elementary properties: It is easy to see that if x_1 is reachable from x_0 and x_2 is reachable from x_1, then:

x_2 is reachable from x_0;

the empty set and the whole space are g.r.z.;
the union of g.r.z. is also a g.r.z.;
the intersection of g.r.z. is also a g.r.z.

Bounded perturbator vector

One of the most natural restrictions (4) is the boundedness in euclidian measure

$$(7) \qquad | w(x, t) |^2 = \sum_1^n [w_i(x, t)]^2 \leq k^2,$$

where the constant k may depend on x but not on t. In this case we have the

THEOREM 2: *Given* (1) *and the condition* (7), *let*

$$(8) \qquad x = \xi(\tau) \qquad \xi(0) = x_0$$

be the equation of an arc of curve contained in a domain where

$$(9) \qquad |f(x)| < k.$$

Then (8) *is the solution of* (1) *with a conveniently chosen* $w(t)$.

THEOREM 3: *If, with the above conditions,*

$$|f(x)| > k$$

at some points of the curve (8), *this curve is a possible path of* (1), *if*

$$(10) \qquad |f(x)|^2 - \left(\frac{d\xi/d\tau \cdot f(x)}{|d\xi/d\tau|}\right)^2 \leq k^2$$

(*the dot means scalar product*).

Intuitively, both theorems are evident by considering that, at a point x, the vector $f(x) + w$ may have any direction in the first case; meanwhile in the second one, the possible directions fill out a certain cone, whose equation is precisely (10) (see Figs. 1 and 2).

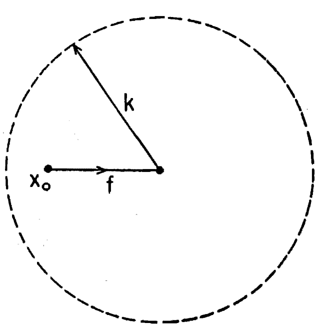

FIG. 1

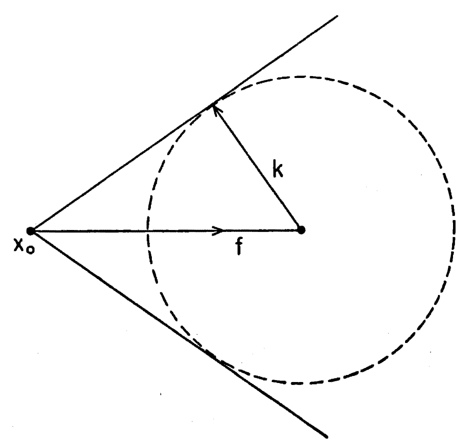

FIG. 2

I shall give then two examples, one in n-space and one in the plane.

Example 1: $x = -\lambda x + w(t)$, with $\lambda > 0$; $| w |^2 \leq k^2$. According to Theorem 1, in the sphere $| x | < k/\lambda$ any sufficiently smooth curve is a possible path. On the surface $| x | = k/\lambda$ the vector $\dot{x}$ is directed to the inside for any choice of w, so that there is no path going from the inside to the outside. From this, we deduce that the interior of this sphere is the reachable zone (r.z.) from any of its points.

If we take a point x_0 outside the sphere $| x | \leq k/\lambda$, the cone of possible directions is just tangent to this sphere, so that the reachable zone from x_0 is that cone plus the sphere (Fig. 3).

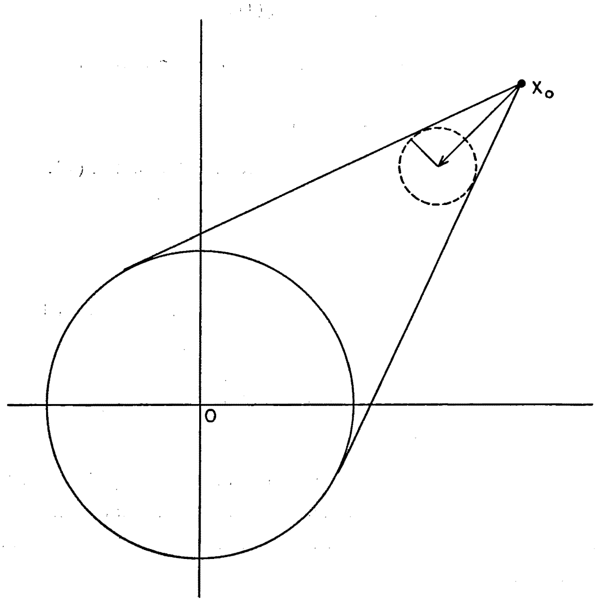

Fig. 3

Example 2:

$$\dot{x}_1 = a_1 x_1 + w_1 \qquad w_1^2 + w_2^2 \leq k^2$$
$$\dot{x}_2 = a_2 x_2 + w_2$$

If $a_1, a_2 > 0$, the origin is an unstable node, and the r.z. shown in Fig. 4 arises. (From the interior of the ellipse, the whole plane is reachable). If $a_1, a_2 < 0$, the origin is a stable node (see Fig. 5). All the linear cases are easily solved in the plane (x_1, x_2), by means of graphical integration.

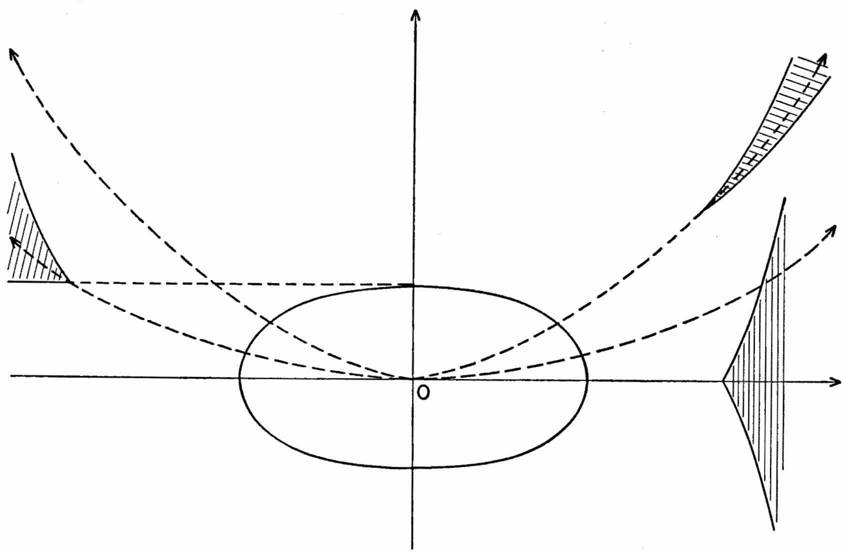

FIG. 4

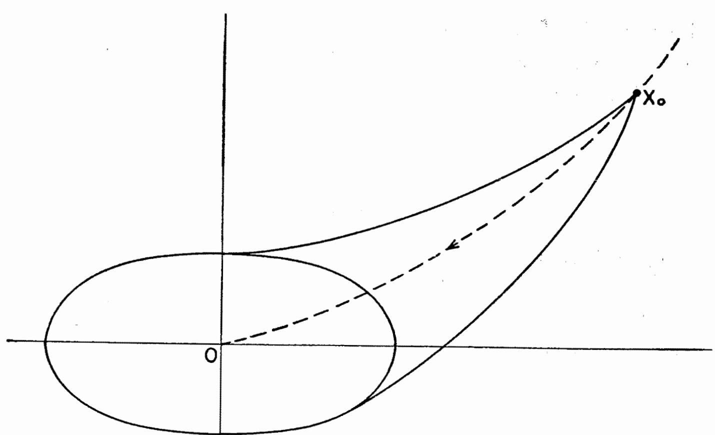

FIG. 5

Other types of boundedness: The most interesting is probably the boundedness of each component independently

(11) $$| w_i | \leq k_i .$$

EXAMPLE 3:

$$\dot{x} = -ax + w \qquad | w_i | \leq k_i$$

This is analogous to Example 1, giving the result shown in Fig. 6.

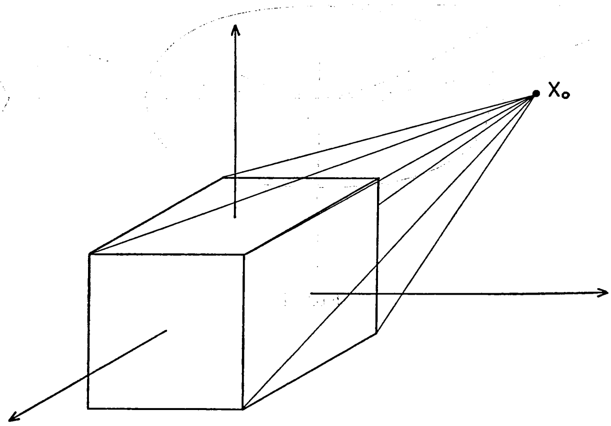

FIG. 6

Perturbator vector with fewer dimensions

Let us take now the first of conditions (4), or the (5) with $s < n$. In this brief explanation I will restrict myself to the 2-dimensional case ($x = x_1 , x_2)$), most of the results being generalizable to the n-space, with suitable additional conditions.

Let us write

(12) $$w(x, t) = \varphi(x) \cdot \alpha(t)$$

being $\varphi(x)$ fixed and $\alpha(t)$ arbitrary. The differential equation (1) is then

(13) $$\dot{x} = f(x) + \varphi(x) \cdot \alpha(t).$$

We decompose this equation in (2)

$$\dot{x} = f(x)$$

the fundamental equation, and

(14) $$\dot{x} = \varphi(x)$$

the perturbator equation, having omitted the factor $\alpha(t)$ (which does not change the solutions-paths).

Equations (2) and (14) define two families of curves, the fundamentals and the perturbatrices. A point x_0 will be called regular if $f(x_0)$ and $\varphi(x_0)$ are linearly independent. Since $f(x)$ and $\varphi(x)$ are continuous in a certain neighbourhood of a regular point, both families of curves intersect at an angle of constant sign ($0 < \beta < \pi$ or $-\pi < \beta < 0$), and are topologically equivalent to the horizontal and vertical lines of a coordinate system. The analysis of the reachable zone in this neighbourhood is then quite simple. In the large, topological complications may arise, but some general results can be established, as follows:

THEOREM 4: *Given the system* (13), *a g.r.z.* Z *and a regular point* x_0 *belonging to the boundary of* Z, *there is a neighbourhood* U *of* x_0 *such that in* U *the boundary of* Z *is exactly the perturbatrix through* x_0.

DEFINITION 4: A point x_1 will be called *quasi-reachable* (from x_0 or from a g.r.z. Z) if in every neighbourhood of x_1 there are reachable points (from x_0 or Z).

THEOREM 5: *Given the system* (13) *and a point* x_0, *every other* x_1 *belonging to the perturbatrix through* x_0 *is quasi-reachable from* x_0.

THEOREM 6: *If* x_1 *is quasi-reachable* (*from* x_0 *or from a g.r.z.* Z), *and* x_2 *is quasi-reachable from* x_1, *then so is* x_2 *from* x_0 (*or* Z).

THEOREM 7: *If* x_0 *is a reachable point from some g.r.z.* Z *which is not on the boundary of* Z, *then every other point* x_1 *of the perturbatrix through* x_0 *is also reachable.*

THEOREM 8: *If* x_0 *is not quasi-reachable* (*from* Z), *any other point* x_1 *of the same perturbatrix cannot be quasi-reachable.*

Crossable and not crossable curves

Given an arc of Jordan curve γ in the x-plane,

$$x = \xi(\tau) \qquad 0 \leq \tau \leq 1 \tag{15}$$

oriented for increasing values of τ, and a differential system

$$\dot{x} = f(x, t), \tag{16}$$

defined in a certain region containing the arc γ, we will now define the case in which γ is *crossable* or *not crossable* by the solutions of (16), in the time interval T_1, T_2. For this purpose, we start joining the end points A, B of the arc γ by another arc γ', which does not intersect γ, so that $\gamma + \gamma'$ form a complete Jordan curve, dividing the x-plane in two regions R_l and R_r (where the subscripts l and r are chosen for left and right, in accordance to the side of the oriented Jordan curve where each region is situated). Of course, γ' has to be contained in the region where (16) is defined. Along the t-axis, we shall consider the interval

$$T = (T_1, T_2) \qquad (T_1 \leq t \leq T_2) \tag{17}$$

and the 3-dimensional regions $R_l \times T$ and $R_r \times T$. With these elements and using the notation

$$x = F(t, x_0, t_0) \tag{18}$$

for the general solution of (16), we are able to give the following

DEFINITION 5: The above mentioned arc γ will be called *crossable from left to right* by the differential field (16) in the time interval T, if the arc γ' can be chosen so that there exists in the 3-dimensional space two points (x_1, t_1) and (x_2, t_2) with the properties:

i) $(x_1, t_1) \in R_l \times T$; $(x_2, t_2) \in R_r \times T$;

ii) $t_2 > t_1$;

iii) $F(t_2, x_1, t_1) = x_2$;

iv) the projection of the path (x_1, t_1), (x_2, t_2) on the x-space does not intersect the arc γ' (including its end points A, B), so that it does intersect the arc γ (excluding its end points A, B).

If this definition holds for some γ', so it does for any other γ'' contained in the region of definition of (16).

If the conditions of the above definition do not apply to the arc γ, we say that γ *is not crossable from left to right* in the interval T by the field (16). Similarly, we define expressions like γ *is crossable from right to left*, γ *is crossable in both senses*, etc.

THEOREM 9: *If γ, γ' which form together a closed Jordan curve, and a path of* (16): $(x_1, t_1; x_2, t_2)$ *fulfill all the conditions of the above definition, with the exception that x_2, instead of belonging to R_r, belongs to γ itself (excluding the end points), then the arc γ is crossable from left to right in the considered interval.*

THEOREM 10: *If the oriented Jordan curve γ is composed by the arcs $\gamma_1, \gamma_2, \cdots, \gamma_n$, any of them being not crossable from left to right by* (16) *in the interval T, then γ divides the plane in the regions R_1, R_r, so that in the interval T there is no path of* (16) *from a point of R_1 to a point of R_r*. The same holds for a Jordan curve on Gauss' sphere (the plane plus the point at infinity).

We consider now the angle β from vector tangent to the curve γ, to the field vector given by (16): We suppose that it is defined and varies continuously, with the exception of a finite number of points of γ.

THEOREM 11: *If at an inner point x_0 of the arc γ (for a value of $t \in T$), the angle β defined above is $0 < \beta < \pi$, then γ is crossable from right to left. Conversely, if at all points of γ, $-\pi \leq \beta \leq 0$ (for all $t \in T$), then γ is not crossable from right to left (during T).*

Analytical condition

In our theory we have to consider as curve γ (possible boundary of a g.r.z.), arcs of perturbatrices, whose tangent vector is $\varphi(x)$. The field vector being $f(x)$, the condition

$$0 \leq \beta \leq \pi$$

is equivalent to

$$F[f, \varphi] = \varphi_1 f_2 - \varphi_2 f_1 \geq 0 \tag{19}$$

where the subscripts mean cartesian components. Here we suppose $\varphi_1^2 + \varphi_2^2 \neq 0$, the curve γ being a perturbatrix.

THEOREM 12: *If on the perturbatrix γ there is a point x_0 such that at x_0*

$$\varphi_1 f_2 - \varphi_2 f_1 > 0 \tag{20}$$

then γ is crossable from right to left by some path of (1), (*for example by* (2)). *Conversely, if γ is crossable from right to left by some path of* (1), *it is crossable by* (2) *and at some point of γ* (20) *holds.*

THEOREM 13: *If the oriented Jordan curve γ is composed by the arcs $\gamma_1, \gamma_2, \cdots, \gamma_n$ which are*

i) *arcs of perturbatrices such that at each one, $\varphi_1 f_2 - \varphi_2 f_1$ has constant sign or equals zero, its sign being such that $0 \leq \beta \leq \pi$ in surveying γ in its preassigned sense;*

ii) *critical lines for equation* (3); *i.e., at all points of this γ_i: $\varphi_1(x) = \varphi_2(x) = 0$ and in this case, also $0 \leq \beta \leq \pi$, then γ is the boundary of a g.r.z. of* (2).

EXAMPLE: $\dot{x}_i = f_i(x_1, x_2) + b_i\alpha(t)$, f_i being homogeneous functions of degree m, and b_i constants.

The only possible boundaries of g.r.z. are the straight perturbatrices $\dot{x}_i = b_i$. The perturbatrix through the origin $x_2/x_1 = b_2/b_1$ satisfies (19) at all of its points if

$$\begin{aligned} b_1 f_2(x_1, x_2) - b_2 f_1(x_1, x_2) &= b_1 x_1^m f_2(1, x_2/x_1) - b_2 x_1^m f_1(1, x_2/x_2) \\ &= x_1^m[b_1 f_2(1, b_2/b_1) - b_2 f_1(1, b_2/b_1)] \end{aligned}$$

does not change its sign; i.e., if m is even. For a perturbatrix not passing through the origin, we rotate the axis and write the system

$$\dot{x}_1 = g_1(x_1, x_2)$$

$$\dot{x}_2 = g_2(x_1, x_2) + \alpha(t)$$

so that this line is the boundary of a g.r.z. if $g_1(1, \mu)$ does not change its sign for $-\infty < \mu < +\infty$

Perturbator vector with fewer dimensions and with a condition of boundedness

In some sense, this is a combination of both types of restrictions studied before. In n-space there are many cases to consider, but in the plane, the problem

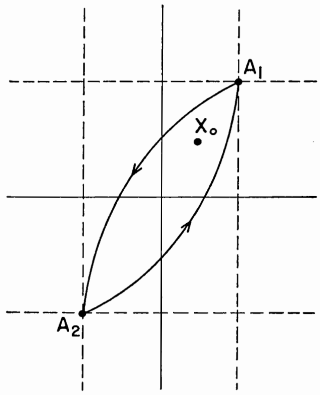

FIG. 7

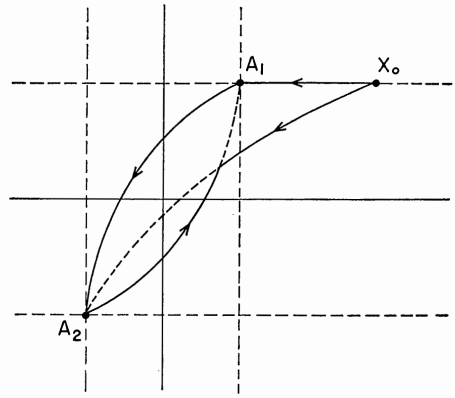

FIG. 8

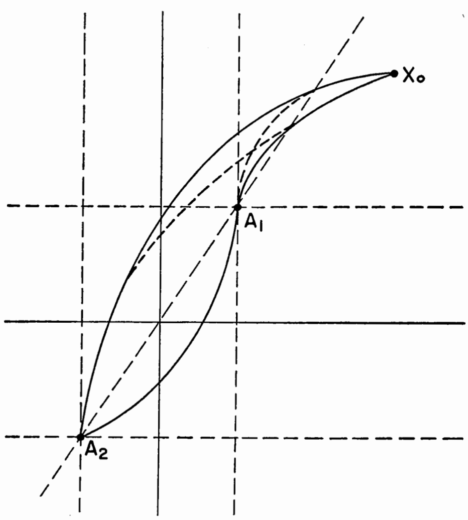

FIG. 9

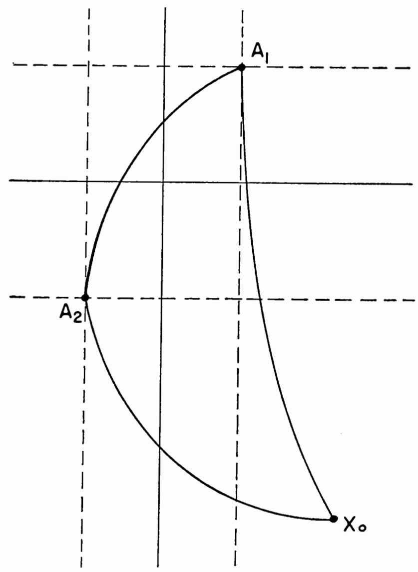

FIG. 10

is simply

(21) $$\dot{x} = f(x) + \varphi(x)\cdot\alpha(t)$$

with $f(x)$, $\varphi(x)$ fixed vectors and $\alpha(t)$ arbitrary, restraint by the condition

(22) $$|\alpha(t)| \leq k.$$

At a certain point x_0, the vector $\dot{x}$ may take, therefore, any direction in the angle limited by $f(x_0) \pm k\varphi(x_0)$.

We may consider also the case of *oriented perturbation*

(23) $$\alpha(t) \geq 0.$$

Here too $\dot{x}$ varies in an angular region, defined by $\dot{x} = f(x_0)$ and $\dot{x} = \varphi(x_0)$. Both cases are, consequently, essentially the same.

To solve this problem, one has to consider the integrals of the *limit cases*

$$\text{(I)}\quad \dot{x} = f(x) + k\varphi(x)$$

$$\text{(II)}\quad \dot{x} = f(x) - k\varphi(x),$$

respectively,

$$\text{(I)}\quad \dot{x} = f(x)$$

$$\text{(II)}\quad \dot{x} = \varphi(x).$$

All the theorems of crossable or not crossable curves are applicable to these cases, and if a curve γ is not crossable by either system I or II, then it is not crossable for any path of (21).

What complicates the situation here is that the boundary of a g.r.z. is not necessarily a solution of one of both systems I, II. If γ is the boundary of a g.r.z. Z, the only thing one can say is that at every regular point x_0, the tangent to γ is exterior to the angle defined by I and II.

Nevertheless, many simple cases are easily solvable, for example, if $f(x)$ is linear and $\varphi(x)$ constant. Some astonishing results are obtained even in these simple cases, i.e.

$$\dot{x}_1 = a_1x_1 + b_1\alpha(t)$$

$$\dot{x}_2 = a_2x_2 + b_2\alpha(t)$$

with $a_2 < a_1 < 0$, b_i constants, $|\alpha(t)| \leq k$, as shown in Figs. 7–10.