CONTROLS: AN APPLICATION OF THE DIRECT METHOD OF LIAPUNOV*

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The stability of systems with nonlinear controls presents one of the most interesting applications of the direct method of Liapunov. This application was first dealt with, we believe, by the Soviet mathematician Lurye [1]. There are noteworthy contributions by Letov [2], Malkin (see [2]) and Yacubovich [3]. The basic treatment of the first two rested upon a transformation of coordinates, and while most ingenious it was assuredly excessively explicit. A treatment by matrices was outlined by Malkin and carried out much further by Yacubovich. A parallel development was also described about the same time by Bass (unpublished material). Of the work of Yacubovich we only know the outline in his short Doklady notes. Our main purpose here is to present and extend the more or less complete argument of these latter authors. Let it be observed in passing that, with modern computing machines, the treatment by matrices is as accessible as any to computation, and is certainly more direct than any other treatment.

Remark. Standard matrix notations are used. An n-vector x with components x_i is thought of as a column-vector ($n \times 1$ matrix). The corresponding row-vector is written x'. The quadratic form $F = \sum a_{ij}x_ix_j$ where $a_{ij} = a_{ji}$, $A = (a_{ij}) = A'$, is written $x' \cdot Ax$. If $F > 0[< 0]$ for $x \neq 0$ we say that F is a positive [negative] quadratic form, written also $A > 0[< 0]$.

A real square matrix whose characteristic roots all have negative real parts is said to be *stable.*

1. The problem. A Liapunov function

A very wide class of control problems leads to a system of the form

$$\text{(F)} \qquad \begin{aligned} \dot{x} &= Ax + f(\sigma)b \\ \dot{\sigma} &= c'x - rf(\sigma) \end{aligned}$$

and this is the system which we propose to discuss. Here x, b, c are n-vectors and $\sigma, f(\sigma)$, r are scalars. The matrix A is constant. The system $\dot{x} = Ax$ is the initial physical system, the components of x are its parameters, while σ, $f(\sigma)$ are the feedback signal and characteristic.

To simplify matters we assume that $f(\sigma)$ is continuous. Moreover, $\sigma f(\sigma) > 0$ for $\sigma \neq 0$ and $\int_0^s f(\sigma)d\sigma \rightarrow \infty$ as $|s| \rightarrow \infty$.

Until further notice and as a working hypothesis the matrix A is assumed to be stable.

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A certain simplication may be made as follows. Let the component b_h of b be $\neq 0$. Then by changing the scale of measurement of x_h one may replace it by $b_h x_h$ and hence b_h by $+1$. The b_h are certainly not all zero since it would mean a non operating control. One may then say that the true control parameters are the components of c and the scalar r.

The problem is *to determine the control so as to obtain asymptotic stability for all initial positions of x and all allowable choices of $f(\sigma)$.*

The solution will consist in finding a function $V(x, \sigma)$, of class C^1 in x, σ which is >0 except for $x = 0$, $\sigma = 0$ when it vanishes and whose time derivative

$$\dot{V} = \frac{d}{dt} V(x(t), \quad \sigma(t))$$

along any solution of (F) is a negative definite function of x and $f(\sigma)$. Moreover $V \to \infty$ as $\| x \|^2 + \sigma^2 \to \infty$. By modification of a well known theorem of Liapunov we will then have the desired kind of stability.

The function that we shall study, due to Lurye, is of the form

$$V(x, \sigma) = x' \cdot Bx + \int_0^\sigma f(\sigma) \, d\sigma. \tag{1.1}$$

It is certainly positive except for $x = 0$, $\sigma = 0$ when it vanishes, provided that $B > 0$. We find at once from (F):

$$-\dot{V}(x, \sigma) = x' \cdot Bx + x' \cdot B\dot{x} + rf^2(\sigma) - 2f(\sigma)\{\tfrac{1}{2}(b' \cdot Bx + x' \cdot Bb) + \tfrac{1}{2}c' \cdot x)\}.$$

Introduce the new vector g whose components are

$$g_k = \sum b_{ki} b_i$$

so that $g = Bb$. Set also

$$A'B + BA = -C. \tag{1.2}$$

Then we may write

$$-\dot{V} = +x' \cdot Cx + rf^2(\sigma) - 2f(\sigma)(g' + \tfrac{1}{2}c') \cdot x. \tag{1.3}$$

We find at once that $C' = C$, so that C is a matrix of a quadratic form. Thus $\dot{V}$ is a quadratic form in $x_1, \cdots, x_n, f(\sigma)$.

Take any symmetric $C > 0$. It is well known ([4]) that, since A is stable, the relation (1.2) has a unique solution as a stable symmetric matrix B, which is given by

$$B = \int_0^{+\infty} e^{A't} C e^{At} \, dt.$$

Under the circumstances $\dot{V}$ is a quadratic form in $x_1, \cdots, x_n, f$. Since $C > 0$ the first n of the well-known conditions to have the form $-\dot{V}$ definite positive are already satisfied. The only condition left (a necessary and sufficient condition) is that the determinant

$$\begin{vmatrix} & C, -\left(g + \frac{1}{2}c\right) \\ -\left(g' + \frac{1}{2}c'\right), r & \end{vmatrix} > 0.$$

Let $| C | = \delta > 0$. From known theorems on determinant expansion (2.3) reduces to

$$\delta r - \delta(g' + \tfrac{1}{2}c') \cdot C^{-1}(g + \tfrac{1}{2}c) > 0$$

or finally to

$$r > (g' + \tfrac{1}{2}c') \cdot C^{-1}(g + \tfrac{1}{2}c). \tag{1.4}$$

Since $C > 0$ likewise $C^{-1} > 0$. Therefore the right hand side of (1.4) $\geqq 0$ and so $r > 0$. However, it should be observed that the control efficiency actually increases with r, since increasing r means moving away from the situation where $\dot{V}(x, \sigma)$ ceases to be negative definite.

An example. Let the characteristic roots $-\mu_1, \cdots, -\mu_n$ of A be real and distinct, hence negative: $\mu_n > 0$. Choose coordinates such that $A = -D$, $D = \operatorname{diag}(\mu_1, \cdots, \mu_n)$. Then the system (F) is

$$\dot{x} = -Dx + f(\sigma)b$$
$$\dot{\sigma} = c' \cdot x - rf(\sigma).$$

Furthermore suppose that no $b_h = 0$ so that $b = (1, 1, \cdots, 1)$. The explicit form of the system (1.2) is here

$$(\lambda_i + \lambda_k)b_{ik} = c_{ik}$$

Choose now $C = \operatorname{diag}(d_1, \cdots, d_n)$, $d_h > 0$, so that

$$C^{-1} = \operatorname{diag}\left(\frac{1}{d_1}, \cdots, \frac{1}{d_n}\right), \quad B = \operatorname{diag}\left(\frac{d_1}{2\mu_1}, \cdots, \frac{d_n}{2\mu_n}\right)$$

and (1.4) becomes

$$r > \sum \frac{1}{4d_h}\left(\frac{d_h}{\mu_h} + c_h\right)^2$$

or

$$r > \sum \frac{1}{4}\left(\frac{e_h}{\mu_h} + \frac{c_h}{e_h}\right)^2, \quad e_h = \sqrt{d_h}.$$

The h-th parenthesis in the sum is least for $e_h^2 = c_h\mu_h$ if $c_h > 0$, or zero if $c_h \leqq 0$. Hence

$$r > \sum \frac{c_h}{\mu_h}$$

where the sum is extended to all h for which c_h is positive. This gives a lower bound, in the present case for the value of r.

2. Control with some characteristic roots zero

Let several (not all) the characteristic roots of A be zero. A comparatively simple situation will arise when, by a suitable choice of coordinates the basic system (F) may be put in the form

$$\begin{aligned} \dot{x} &= Ax + f(\sigma)b, \\ \dot{y} &= f(\sigma)\, d, \\ \dot{\sigma} &= c'\cdot x + 2e'\cdot y - rf(\sigma). \end{aligned}$$

Here the notations are so chosen that x, b, c are n-vectors, A is the same as before, and y, d, e are p-vectors. In particular, the components of b, d are 0 or 1. As for $\sigma, f(\sigma), r$ they are still scalars. Since the y_h remain fixed in the uncontrolled system they are known as *neutral* parameters.

We look for a Liapunov function

$$V(x, y, \sigma) = y'\cdot My + \left\{x'\cdot Bx + \int_0^\sigma f(\sigma)\, d\sigma\right\},$$

the $\{\cdots\}$ being the same as previously and $M > 0$. By direct calculation we find

$$-\dot{V} = x'\cdot Cx + rf^2(\sigma) - 2f(\sigma)(g' + \tfrac{1}{2}c')\cdot x + (d'M + e')f(\sigma)\cdot y,$$
$$A'B + BA = -C.$$

If one chooses M and d so that

$$Md + e = 0 \tag{2.1}$$

and treat the remaining terms as done earlier, one will obtain a Liapunov function V such that $-\dot{V} > 0$ for x, $\sigma \neq 0$ but zero for x, $\sigma = 0$, $y \neq 0$. Hence the Liapunov function under consideration only guarantees stability but not asymptotic stability in the large. By a further argument it is possible to show that every solution approaches the linear subspace of critical points defined by $x = 0$, $\sigma = 0$ and $e'\cdot y = 0$ as $t \to \infty$.

To proceed a little further suppose that $d = (1, 1, \cdots, 1)$. Upon setting

$$m_1 = \sum m_{ij}, \quad m' = (m_1, \cdots, m_p), \tag{2.2}$$

the system (2.1) yields

$$m = -e. \tag{2.3}$$

The question arises then whether one may determine $M > 0$ such that (2.3) holds.

Consider first the case $p = 2$. We have then

$$m_{11} + m_{12} = -e_1, \quad m_{12} + m_{22} = -e_2$$

and since $M > 0$:

$$m_{11}m_{22} - m_{12}^2 > 0.$$

For simplicity set $m_{12} = \rho$. Then $m_{11} = -e_1 - \rho$, $m_{12} = -e_2 - \rho$, and ρ is subject to the sole inequality

$$(-e_1 - \rho)(-e_2 - \rho) - \rho^2 = e_1e_2 + (e_1 + e_2)\rho > 0.$$

If $e_1 + e_2 = 0$ this becomes $e_1e_2 > 0$ which is ruled out. If $e_1 + e_2 \neq 0$ one may always choose a suitable ρ.

The general case is, of course, more complicated. Let us endeavor to choose

$$M = \begin{pmatrix} \Delta & \mu \\ \mu' & s \end{pmatrix}$$

where $\Delta = \text{diag}(\rho, \cdots, \rho)$ ($p - 1$ terms), $\mu' = (\mu_1, \cdots, \mu_{p-1})$ and s is a scalar. We assume $\rho > 0$. Thus $\Delta > 0$ and the condition for M to be >0 is (same calculation as earlier)

$$s > \sum \frac{\mu_i^2}{\rho}.$$

On the other hand the relations (2.3) are here

$$\mu_i + \rho = -e_i, \ (i < p); \quad s + \sum \mu_i = -e_\mu .$$

Upon setting

$$\textstyle\sum_{i<p} e_i = h, \quad \sum_{i<p} e_1^2 = k > 0,$$

a simple calculation reduces (2.4) to

$$\nu = h\rho + k + e_p < 0. \tag{2.5}$$

If the line $\nu = h\rho + k + e_p$ in the (ρ, ν) plane intersects the positive ρ axis, one may select ν to satisfy (2.5), and hence choose M in the special form selected. However, if this special form is not admissible, it may very well be that one may, nevertheless, select M to satisfy (2.3) and $b > 0$.

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BIBLIOGRAPHY

[1] A. I. LURYE. *Some nonlinear problems in the theory of automatic regulation*, Gosizdat. Techniko-teoret. lit., Moscow, 1951.

[2] A. M. LETOV. *Stability of nonlinear regulating systems*, *ibid.*, 1955.

[3] V. A. YACUBOVICH. Doklady, 117 (1957) 44–46; 121 (1958), 984–986.

[4] RICHARD BELLMAN. *Kronecker products and the second method of Liapunov*, Math. Nachrichten, 20 (1959) 17–19.