PARALLELIZABLE FLOWS AND LYAPUNOV'S SECOND METHOD

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This is an abstract of the paper [1] which is to appear elsewhere.

Let $\mathfrak{F}$ be a flow on a metric space R, given by a continuous map $f:R \times E \to R$ for which $f(p, 0) = p$ and $f(f(p, t), t') = f(p, t + t')$ whatever $p \in R$ and $t, t' \in E$. We say that $\mathfrak{F}$ is parallelizable on R if there exists a set $S \subset R$ and a homeomorphism $h:R \to S \times E$ such that $\{f(p, t):p \in S, t \in E\} = R$ and $h(f(p, t)) = (p, t)$ for every $(p, t) \in S \times E$.

Clearly, if $\mathfrak{F}$ is parallelizable on R then S is a section of $\mathfrak{F}$ in the sense that to every $p \in R$ there corresponds a unique $t_S(p) \in E$ for which $f(p, -t_S(p)) \in S$; moreover, t_S is continuous on R and $p \to f(p, -t_S(p))$ is a deformation retract of R onto S. Also, a parallelizable flow is obviously dispersive [6]: for every distinct $p, q \in R$ there exist neighborhoods $U, V \subset R$ and $I = (-T, T) \subset E$ such that $f(U, t) \cap V = \emptyset$ for all $t \in' I$.

The principal results of Part I of [1] are these: A flow $\mathfrak{F}$ on R is parallelizable on R if and only if $\mathfrak{F}$ has a section S with t_S continuous on R; if R is in addition locally compact and separable, this is the case if and only if $\mathfrak{F}$ is dispersive.

Our proof of the latter result, which is closely related to a theorem of Barbašin [2], proceeds as follows.

We show first that if R is locally compact and $p_0 \in R$ is a wandering point of $\mathfrak{F}$ there exists a compact set S_0 containing p_0 which is a section of the flow on the tube $T(S_0) \equiv f(S_0, E)$ with t_{S_0} continuous on $T(S_0)$. Next we prove that if $\mathfrak{F}$ is dispersive $T(S_0)$ is closed in R. This fact appears to be crucial. Given any two such tubes $T(S_0)$, $T(S_0')$ in a dispersive flow with $T(S_0) \cap S_0' \neq \emptyset$, it permits us to construct a compact set S_0'' such that $T(S_0'') = T(S_0) \cup T(S_0')$ and t_{S_0}'' is continuous on $T(S_0'')$. Intuitively, we do this by simply sliding S_0' along $T(S_0')$ to join S_0. Thus, if $\mathfrak{F}$ is dispersive and R is in addition separable, we obtain by a simple induction a closed (not necessarily compact) set $S \subset R$ which is a section of $\mathfrak{F}$ with t_S continuous on R.

As a by-product, we show that for locally compact separable metric spaces Niemytskii's concept [7] of a completely unstable flow with no improper saddle point coincides with that of a dispersive flow.

Part II of [1] concerns constructions of Lyapunov functions for differential equations (*) $\dot{x} = X(x, t)$ with X continuous and sufficiently smooth on $E^n \times E$ and $X(0, t) = 0$ on E, when $x = 0$ is uniformly stable or uniform-asymptotically stable or unstable. For these cases we prove the converses to the classical theorems of Persidskii, Lyapunov, and Četaev (cf., e.g., [3]) by considering the dispersive flow $\mathfrak{F}_t$ on $E^n \times E$ defined by the trajectories of (*).

Our method of construction is this. We wish to find on a cylinder $C = B(0, r) \times E$ a real valued continuous function V with specified behavior on C and along the trajectories of (*). Instead of dealing with complicated trajectories and simple bounding surfaces we use the parallelizability of $\mathfrak{F}_t$ to transform the problem into one concerning straight line trajectories and complicated boundaries. In this manner the behavior of V along the trajectories is trivial to control, and the other desired properties of V are consequences of the existence of suitable sections in the parallel flow. This brings into sharp focus the geometric picture of the particular situation at hand, which is not present in previous entirely analytical constructions and which is obscured by the details of the constructions Krasovskii [4, 5] used in applying Barbašin's method of sections [2] to the original flow $\mathfrak{F}_t$. It also serves to give a simple unified approach to the constructions of Lyapunov functions for all converse theorems.

An interesting consequence of our method is the fact that to obtain Lyapunov functions of the same class as the differential equation requires essentially no considerations different from those involved in constructing continuous Lyapunov functions; one simply needs to appeal to Whitney's instead of Tietze's extension theorem.

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