LIAPUNOV'S FUNCTION AND BOUNDEDNESS OF SOLUTIONS*

By Taro Yoshizawa

Liapunov has discussed the stability of solutions of a system of differential equations by utilizing a scalar function satisfying some conditions [3]. And the relations between Liapunov functions and various types of stability have been discussed by many authors. For boundedness as well as stability, Liapunov's theory is very useful and the relations between Liapunov functions and various types of boundedness are very similar to those between Liapunov functions and various types of stability.

Now we consider a system of differential equations,

$$\frac{dx}{dt} = F(t, x), \tag{1}$$

where x denotes an n-dimensional vector and $F(t, x)$ is a given vector field which is defined and continuous in the domain

$$\Delta : 0 \leqq t < \infty, \ \| x \| < \infty \quad \text{(the norm is the Euclidean norm).}$$

Let $x = x(t; x_0, t_0)$ be a solution of (1) through the initial point (t_0, x_0). Unless otherwise stated, we consider the solution for $t \geqq t_0$.

There are various types of boundedness [6], but now we consider the following types.

Definitions. (i) The solutions of (1) are said to be uniformly bounded, if for any $\alpha > 0$ there exists a positive number β such that if $\| x_0 \| \leqq \alpha$, $\| x(t; x_0, t_0) \| < \beta$ for $t \geqq t_0$, where β depends only on α and is independent of t_0.

(ii) The solutions of (1) are said to be equiultimately bounded for the bound B, if there exist positive numbers B and T such that $\| x(t; x_0, t_0) \| < B$ for $t > t_0 + T$, where if $\| x_0 \| \leqq \alpha$, T is determined depending only on t_0 and α. B is independent of the particular solution.

(iii) The solutions of (1) are said to be uniform-ultimately bounded for the bound B, if T in (ii) is determined depending only on α and independent of t_0.

If (1) is a linear homogeneous system, the stability of $x(t) \equiv 0$ and the boundedness are equivalent [2, 6]; in particular, uniform boundedness is equivalent to uniform stability. And, if $F(t, x)$ in (1) is periodic in t, (ii) and (iii) are equivalent.

When a function $f(t, x)$ satisfies locally the Lipschitz condition with respect to x, we represent this fact by $f(t, x) \in C_0(x)$. Moreover, if $f(t, x)$ satisfies locally

* This research was partially supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract Number AF 49(639)-382. Reproduction in whole or in part is permitted for any purpose of the United States Government.

the Lipschitz condition with respect to (t, x), we represent by $f(t, x) \in C_0(t, x)$.

Now we consider the Liapunov function $V(t, x)$. We assume that $V(t, x)$ is continuous and non-negative in its domain of definition and that $V(t, x) \in C_0(x)$. Corresponding to $V(t, x)$ we define the function

$$V'(t, x) = \overline{\lim_{h \to +0}} \frac{1}{h} \{V(t + h, x + hF(t, x)) - V(t, x)\}.$$

To simplify the statements, we give here some definitions. We will say briefly $V(t, x)$ has the property A when there exists a positive continuous increasing function $a(r)$ such that $V(t, x) \leqq a(\| x \|)$. We will say $V(t, x)$ has the property B when there exists a non-negative continuous increasing function $b(r)$ such that $b(\| x \|) \leqq V(t, x)$ and $b(r) \to \infty$ as $r \to \infty$. Moreover we will say $V(t, x)$ has the property C when there exists a positive continuous function $c(r)$ such that $V'(t, x) \leqq -c(\| x \|)$. Let Δ^* be a domain defined by $0 \leqq t < \infty$, $\| x \| \geqq R_0$ (R_0 may be sufficiently large).

THEOREM 1. *If there exists a positive Liapunov function $V(t, x)$ which is defined in Δ^* and has the properties A, B and if we have $V'(t, x) \leqq 0$ in the interior of Δ^*, the solutions of* (1) *are uniformly bounded.*

PROOF. For any $\alpha > 0$ (we may suppose $\alpha > R_0$), we have $V(t, x) \leqq a(\alpha)$ when $\| x \| = \alpha$ by the property A. By the property B, we can choose β so that $b(\beta) > a(\alpha)$. Now we suppose that for some solution $x = x(t; x_0, t_0)$ issuing from (t_0, x_0) $(0 \leqq t_0 < \infty, \| x_0 \| \leqq \alpha)$, we have $\| x(t'; x_0, t_0) \| = \beta$ at some t, say t'. Then there exist t_1 and t_2 such that $\| x(t_1; x_0, t_0) \| = \alpha$, $\| x(t_2; x_0, t_0) \| = \beta$ and $\alpha < \| x(t; x_0, t_0) \| < \beta$ for $t_1 < t < t_2$. Considering the function $V(t, x(t; x_0, t_0))$, we have $V(t_1, x(t_1; x_0, t_0)) \leqq a(\alpha)$ and $V(t_2, x(t_2; x_0, t_0)) \geqq b(\beta)$. On the other hand, we have $V(t_2, x(t_2; x_0, t_0)) \leqq V(t_1, x(t_1; x_0, t_0))$ by the condition $V'(t, x) \leqq 0$, whence we have $b(\beta) \leqq a(\alpha)$ and there arises a contradiction. Therefore if $\| x_0 \| \leqq \alpha$, $\| x(t; x_0, t_0) \| < \beta$; that is to say, the solutions are uniformly bounded.

EXAMPLE: (Antosiewicz [1]). In the equation $x'' + \phi(x, x')x' + h(x) = e(t)$, we suppose that $\phi(x, x')$ and $h(x)$ are continuous for all values of their variables and $e(t)$ is continuous. If $\phi(x, x') \geqq 0$ for all x, x', $H(x) = \int_0^x h(u)\, du > 0$ for all $x \neq 0$, $H(x) \to \infty$ with $|x|$, and $\int^\infty |e(t)|\, dt < \infty$, then every solution satisfies $|x(t)| < c_1$, $|x'(t)| < c_2$ as $t \to \infty$.

In this case we consider the system

$$x' = y, \quad y' = -\phi(x, y)y - h(x) + e(t).$$

If we put

$$V(t, x, y) = \sqrt{y^2 + 2H(x)} - \int_0^t |e(t)|\, dt,$$

this $V(t, x, y)$ satisfies the condition in Theorem 1. Therefore the solutions of this equation are uniformly bounded.

When $F(t, x) \in C_0(x)$, the converse of Theorem 1 is valid [6].

THEOREM 2. *We assume that $F(t, x) \in C_0(x)$. If the solutions of* (1) *are uniformly bounded, there exists a positive Liapunov function $V(t, x)$ defined in Δ^* such that it has the properties A, B and $V'(t, x) \leqq 0$ in the interior of Δ^*. In this case we can obtain $V(t, x)$ such that $V(t, x) \in C_0(t, x)$.*

PROOF. If we put

$$(2) \qquad V(t, x) = \min_{\tau} [\| x(\tau; x, t)\|; \quad \tau \in [0, t] \cap D],$$

where D is the largest interval to the left of t on which $x(\tau; x, t)$ is defined, it is clear that we can define $V(t, x)$ for each point (t, x) in Δ^*. From (2) we can easily see that $V(t, x) \leqq \| x \|$, i.e., $V(t, x)$ has the property A. By the uniform boundedness of solutions, we have $\| x(t; x_0, t_0)\| < \beta(\alpha)$ when $\| x_0 \| \leqq \alpha$. We may assume that $\beta(\alpha)$ is a continuous strictly monotone increasing function of α. Then there exists a function $\alpha(\| x \|)$ such that $0 < \alpha(\| x \|) \leqq V(t, x)$, where $\alpha(\beta)$ is the inverse function of $\beta(\alpha)$ and $\alpha(\beta)$ is a continuous strictly monotone increasing function of β and $\alpha(\beta) \longrightarrow \infty$ as $\beta \longrightarrow \infty$. Thus $V(t, x)$ has the property B.

Since $\| x(\tau; x, t)\|$ takes its minimum at some τ, we can see that $V(t, x) \in C_0(t, x)$. Moreover we have $V'(t, x) \leqq 0$, because for any solution $x = x(t; x_0, t_0)$, $V(t, x(t; x_0, t_0))$ is a non-increasing function of t.

Therefore this $V(t, x)$ is the desired function.

When $F(t, x) \in C_0(t, x)$, we can obtain a Liapunov function $V(t, x) \in C_\infty$ by Massera's method in [5]($V(t, x) \in C_\infty$ means that $V(t, x)$ has continuous partial derivatives of all orders).

THEOREM 3. *If there exists a positive Liapunov function $V(t, x)$ defined in Δ^* and having the properties A, B and C, the solutions of* (1) *are uniform-ultimately bounded. Moreover, in this case, the solutions are uniformly bounded.*

PROOF. By the property B, choosing R_0 suitably, we have a positive number c such that $c \leqq V(t, x)$ for $\| x \| \geqq R_0$. Since the solutions are uniformly bounded by Theorem 1, there is a positive number B such that if $\| x_0 \| \leqq R_0$, $\| x(t; x_0, t_0)\| < B$. Now we consider $x = x(t; x_0, t_0)$ such that $\| x_0 \| \leqq \alpha$, where α is an arbitrary positive number and $\alpha > R_0$. Then there exists a positive number β depending only on α such that $\| x(t; x_0, t_0)\| < \beta$ for $t \geqq t_0$. Considering $V(t, x)$ in the domain, $0 \leqq t < \infty$, $R_0 \leqq \| x \| \leqq \beta$, there exists a positive number λ depending on β such that $V'(t, x) \leqq -\lambda(\beta)$ by the property C. If we suppose that the solution satisfies always $R_0 < \| x(t; x_0, t_0)\| \leqq \beta$ if $R_0 < \| x_0 \| \leqq \alpha$, we have

$$V(t, x(t; x_0, t_0)) - V(t_0, x_0) \leqq -\lambda(t - t_0).$$

From this, we can see that at some t, say t', we have $\| x(t'; x_0, t_0)\| = R_0$, where $t_0 \leqq t' \leqq t_0 + T$ and $T = (a(\alpha) - c)/\lambda$. Hence we have $\| x(t; x_0, t_0)\| < B$ when $t > t_0 + T$, and this T depends only on α. Therefore the solutions of (1) are uniform-ultimately bounded.

When $F(t, x) \in C_0(x)$, we can obtain the converse of Theorem 3 [6]. Namely,

THEOREM 4. *We assume that $F(t, x) \in C_0(x)$. In order that the solutions of (1) are uniformly bounded and uniform-ultimately bounded, it is necessary and sufficient that there exists a positive Liapunov function $V(t, x) \in C_0(t, x)$ defined in Δ^* which has the properties A, B and C.*

If we put

$$V(t, x) = \sup_{\tau} \left[\| x(t + \tau; x, t) \| \frac{1 + \delta\tau}{1 + \tau}; \quad \tau \geqq 0 \right], \tag{3}$$

this $V(t, x)$ is the desired function, where $\delta > 1$ and $\| x \| > \delta B$.

When we apply the theorems on boundedness to differential equations of the second order, we can obtain existence theorems for periodic solutions by using Massera's theorem [4].

For example, consider the equation

$$x'' + kf(x)x' + g(x) = kp(t) \quad (k > 0), \tag{4}$$

where $f(x)$ and $g(x)$ are continuous. We put $P(t) = \int_0^t p(\tau)\, d\tau$ and $F(x) = \int_0^x f(\xi)\, d\xi$ and we assume that (a) $P(t)$ is bounded, (b) $F(x) \to \pm \infty$ as $x \to \pm \infty$ and (c) $xg(x) > 0$ for $|x| \geqq x_0 > 0$ and $G(x) = \int_0^x g(\xi)\, d\xi \to \infty$ as $|x| \to \infty$. Then considering the equivalent system

$$x' = y + kP(t) - kF(x), \quad y' = -g(x)$$

and choosing p and $q > 0$ suitably, we define $V(t, x, y)$ as follows:

$$V(t, x, y) = \begin{cases} G(x) + \dfrac{y^2}{2} & (x \geqq q, |y| < \infty) \\[2ex] G(x) + \dfrac{y^2}{2} - x + q & (|x| \leqq q, y \geqq p) \\[2ex] G(x) + \dfrac{y^2}{2} + 2q & (x \leqq -q, y \geqq p) \\[2ex] G(x) + \dfrac{y^2}{2} + \dfrac{2q}{p}y & (x \leqq -q, |y| \leqq p) \\[2ex] G(x) + \dfrac{y^2}{2} - 2q & (x \leqq -q, y \leqq -p) \\[2ex] G(x) + \dfrac{y^2}{2} + x - q & (|x| \leqq q, y \leqq -p). \end{cases}$$

Then this $V(t, x, y)$ satisfies the condition in Theorem 3, and hence we can see that the solutions are uniform-ultimately bounded. Therefore if $g(x) \in C_0(x)$ and $p(t)$ is periodic, (4) has at least one periodic solution.

For equiultimate boundedness we have the following theorem [6].

THEOREM 5. *We assume that* $F(t, x) \in C_0(x)$. *In order that the solutions of* (1) *are equiultimately bounded, it is necessary and sufficient that there exist a positive number* B *and a non-negative Liapunov function* $V(t, x) \in C_0(t, x)$ *satisfying the following conditions in* Δ; *namely,*

(1) $\alpha(\| x \|) \leqq V(t, x)$ *for* $\| x \| \geqq B$, *where* $\alpha(\tau)$ *is a continuous function which is positive increasing for* $\tau > B$ *and* $\alpha(\tau) \to \infty$ *as* $\tau \to \infty$,

(2) $V'(t, x) \leqq -cV(t, x)$, *where* c *is a positive constant.*

PROOF. We will show that the condition is sufficient. For the sufficient condition, we need not require $F(t, x) \in C_0(x)$. Now if we put

$$W(t, x) = e^{ct}V(t, x),$$

$W(t, x)$ satisfies the following conditions:

(a) $\alpha(\| x \|)e^{ct} \leqq W(t, x)$ for $\| x \| \geqq B$,

(b) $W(t, x) \in C_0(t, x)$,

(c) $W'(t, x) \leqq 0$.

We suppose that for some solution, say $x(t; x_0, t_0)(\| x_0 \| \leqq K)$, we have $\| x(t_m; x_0, t_0)\| > \bar{B}(\bar{B} > B)$ for some sequence $\{t_m\}$ for which $t_m \to \infty$ with m. Then we have

$$W(t_m, x(t_m; x_0, t_0) \geqq \alpha(\bar{B})e^{ct_m}.$$

On the other hand, by the conditions (b) and (c), we have

$$W(t_m, x(t_m; x_0, t_0)) \leqq W(t_0, x(t_0; x_0, t_0)).$$

If we put $\max_{\| x \| \leqq K} W(t_0, x) = \beta(t_0)$, we have $\alpha(\bar{B})e^{ct_m} \leqq \beta(t_0)$. Since $\alpha(\bar{B}) > 0$ and $t_m \to \infty$ with m, there arises a contradiction. Therefore we can see that the solutions are equiultimately bounded for the bound $\bar{B}$.

Next we will show that the condition is necessary. We suppose that the solutions are equiultimately bounded for the bound B. Now we consider a function as follows; namely,

$$G(\zeta) = \begin{cases} \zeta - B & (\zeta \geqq B) \\ 0 & (0 \leqq \zeta < B). \end{cases}$$

And we put

$$V(t, x) = \sup_{\tau} [G(\| x(t + \tau; x, t)\|)e^{\tau};\ \ 0 \leqq \tau]. \tag{5}$$

Then we can see that this $V(t, x)$ is the desired function.

When the solutions of (1) are uniformly bounded and uniform-ultimately bounded, we can find $\beta(\| x \|)$ and $\mu(\| x \|)$ such that $\| x(\tau; x, t)\| \leqq \beta(\| x \|)$, $T(t, x) \leqq \mu(\| x \|)$. Therefore we can see by (5) that we have

$$V(t, x) \leqq G(\beta(\| x \|)e^{\mu(\| x \|)}.$$

From this we can also obtain Theorem 4.

By using Theorem 5, we can obtain a necessary and sufficient condition for equiasymptotic stability in the large [6].

In this case we assume that $F(t, 0) \equiv 0$ for $0 \leqq t < \infty$.

DEFINITION. The solution $x(t) \equiv 0$ is said to be equiasymptotically stable in the large, if $x(t) \equiv 0$ is stable and there exists a positive number $T(t_0, \alpha, \epsilon)$, defined for any $\epsilon > 0$ and any non-negative value of α and $t_0 \geqq 0$, such that $\| x_0 \| \leqq \alpha$, $t_0 \geqq 0$ and $t > t_0 + T(t_0, \alpha, \epsilon)$ imply $\| x(t; x_0, t_0) \| < \epsilon$.

THEOREM 6. *We assume that $F(t, x) \in C_0(x)$. In order that the solution $x(t) \equiv 0$ is equiasymptotically stable in the large, it is necessary and sufficient that there exists a Liapunov function $V(t, x)$ satisfying the following conditions in Δ; namely,*

(1) $V(t, 0) \equiv 0$ *and* $V(t, x) > 0$, *if* $\| x \| \neq 0$,

(2) $\lambda(\| x \|) \leqq V(t, x)$, *where* $\lambda(u)$ *is a continuous increasing function such that* $\lambda(u) > 0$ *for* $u > 0$ *and* $\lambda(u) \to \infty$ *with* u,

(3) $V'(t, x) \leqq -cV(t, x)$, *where* c *is a positive constant.*

In this case, the solutions are equiultimately bounded for any positive number ϵ. Hence for $\epsilon = 1/n (n = 1, 2, \cdots)$, we can define $V_n(t, x)$ in the same way as in Theorem 5. Choosing suitable constants g_n, if we put

$$V(t, x) = \sum_{n=1}^{\infty} \frac{1}{2^n} g_n V_n(t, x),$$

this $V(t, x)$ is the desired function.

RIAS, BALTIMORE, MARYLAND AND NIHON UNIVERSITY, TOKYO, JAPAN

BIBLIOGRAPHY

[1] H. A. ANTOSIEWICZ, *On non-linear differential equations of the second order with integrable forcing term*, J. London Math. Soc., 30 (1955), 64–67.

[2] S. LEFSCHETZ, *Differential equations: Geometric theory*, Pure and Applied Math., Vol. 6, 1957.

[3] A. M. LIAPOUNOFF, *Problème général de le stabilité du mouvement*, Annals of Math. Studies No. 17, 1949.

[4] J. L. MASSERA, *The existence of periodic solutions of systems of differential equations*, Duke Math. J., 17 (1950), 457–475.

[5] J. L. MASSERA, *Contributions to stability theory*, Ann. of Math., 64 (1956), 182–206.

[6] T. YOSHIZAWA, *Liapunov's function and boundedness of solutions*, Funkcialaj Ekvacioj, 2 (1959), 95–142.