GLOBAL ASYMPTOTIC STABILITY FOR NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS AND APPLICATIONS TO REACTOR DYNAMICS

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In this paper, which will appear in complete form elsewhere, conditions are obtained under which all solutions of certain real nonlinear systems of differential equations tend to zero as $t \to \infty$. This study originated in some problems of reactor dynamics; however, the systems investigated are of a quite general nature. Therefore, the results are first given in an abstract setting and then interpreted physically.

Theorem 1 is concerned with the following general form of the Liénard equation

$$\ddot{x} + h(t, x, \dot{x})\dot{x} + f(x) = e(t) \quad (\dot{} = d/dt). \tag{1}$$

THEOREM 1. *Let the following conditions be satisfied*:

(2) *$h(t, x, z), f(x), e(t)$ are sufficiently smooth for a local existence and uniqueness theorem to hold for (1) on $0 \leqq t < \infty; -\infty < x, z < \infty$.*

There exists a constant $k > 0$ such that

$$k \leqq h(t, x, z) \quad (0 \leqq t < \infty; -\infty < x, z < \infty). \tag{3}$$

Given any constant $B > 0$ there exists a constant $K_B > 0$ such that

$$h(t, x, z) \leqq K_B \quad (0 \leqq t < \infty; |x|, |z| \leqq B). \tag{4}$$

Given any constant $B > 0$ there exists a constant $K_B > 0$ such that

$$|f(x_1) - f(x_2)| \leqq K_B |x_1 - x_2| \quad (|x_1|, |x_2| \leqq B). \tag{5}$$

$$xf(x) > 0 \quad (x \neq 0). \tag{6}$$

$$g(x) = \int_0^x f(\xi)\, d\xi \to \infty \text{ as } |x| \to \infty. \tag{7}$$

There exists a constant $K > 0$ such that

$$|e(t)| \leqq K \quad (0 \leqq t < \infty). \tag{8}$$

$$\int_0^\infty |e(t)|\, dt < \infty. \tag{9}$$

Then given any x_0, $\dot{x}_0$ the solution $x(t)$ of (1) satisfying $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ exists on $0 = t < \infty$ and

$$\lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} \dot{x}(t) = 0. \tag{10}$$

An immediate consequence of Theorem 1 is

COROLLARY 1. *If in addition to the hypothesis of Theorem* 1 *the condition* $e(t) \to 0$ *as* $t \to \infty$ *is also satisfied, then* $\ddot{x}(t) \to 0$ *as* $t \to \infty$.

The uniqueness hypothesis (in all the theorems) is not really necessary and is used only to simplify various statements.

If $e(t) \equiv 0$, then Theorem 1 may be thought of as a "global" asymptotic stability theorem of the trivial solution $x(t) \equiv 0$. That is, all solutions tend to zero as $t \to \infty$ and not merely those for which $|x_0|$, $|\dot{x}_0|$ are sufficiently small. If $e(t) \not\equiv 0$, then $x(t) \equiv 0$ is, of course, no longer a solution of (1). This sort of complication of stability problems has been considered, for example, in [1, ch. 13].

As part of the novelty of Theorem 1 lies in the fact that h is allowed to depend on t, it is of interest to focus some attention on hypotheses (3) and (4) which relate to this dependence. That (3) cannot be dropped entirely is obvious from the example $\ddot{x} + x = 0$. Assumption (3) cannot be replaced by $h > 0$; for, if $h(t) \geqq 0$ and $h(t) \in L_1(0, \infty)$, then it can be shown that there exist solutions of

$$\ddot{x} + h(t)\dot{x} + x = 0 \tag{11}$$

which do not tend to zero as $t \to \infty$. Although (4) is automatically satisfied in case h does not depend on t explicitly, it still cannot be dropped entirely. For if $\dot{h}(t) \in C(0, \infty)$, $h(t) > 0$, $h^{-1}(t) \in L_1(0, \infty)$ and $[\dot{h}(t) + 1]h^{-2}(t) \in L_1(0, \infty)$, then there exist solutions of (11) which do not tend to zero as $t \to \infty$. These remarks are established by methods which are in the spirit of this paper; in connection with the last one see [2, p. 137].

The proof of Theorem 1 is subdivided into three parts. These establish (under an increasing number of hypotheses): I, the existence and boundedness (the bound depending on x_0, $\dot{x}_0$) of $x(t)$, $\dot{x}(t)$ on $0 \leqq t < \infty$; II, $\dot{x}(t) \to 0$ as $t \to \infty$; and III, $x(t) \to 0$ as $t \to \infty$. I, which is of interest in itself, is due to Antosiewicz [3] in the special case that h is independent of t. The methods employed involve the same sort of considerations as in the Liapounov second method. Equation (1) s written equivalently as

$$\dot{x} = -z, \quad \dot{z} = -h(t, x, -z)z + f(x) - e(t), \tag{12}$$

and the energy function

$$E(x, z) = g(x) + \tfrac{1}{2}z^2 \tag{13}$$

is introduced together with its total derivative with respect to (1); that is,

$$E'(t, x, z) = -h(t, x, -z)z^2 - e(t)z. \tag{14}$$

Because of the $e(t)z$ term in (14) and because a global rather than a local result is desired, one cannot simply cite the classical Liapounov ordinary stability theorem (see [4, p. 113]) in order to obtain I. The present proof of I is similar to that of [3]. Because $E'(t, x, z)$ is not negative definite in x, z (even for the case $e(t) \equiv 0$), the classical Liapounov asymptotic stability theorem (see

[4, p. 114]) cannot be applied here to give even a local asymptotic result. The proofs of II and III circumvent these difficulties.

Theorem 2 is concerned with the system

$$\begin{aligned} \dot{x} &= -\sum_{i=1}^{n} a_i z_i \\ \dot{z}_i &= -h_i(t, x, z)z_i + b_i f(x) + e_i(t) \quad (i = 1, \cdots, n), \end{aligned} \tag{15}$$

where the a_i and b_i are constants and $z = (z_1, \cdots, z_n)$, which for $n = 1$ is essentially (1).

THEOREM 2. *Let the functions h_i, f, e_i be sufficiently smooth for a local existence and uniqueness theorem to hold for* (15) *on $0 \leqq t < \infty$; $-\infty < x, z_i < \infty$. Let the h_i satisfy* (3, 4); *the f,* (5, 6, 7); *and the e_i,* (8, 9). *Furthermore, let the constants a_i, b_i satisfy either*

$$a_i = c\, b_i, \text{ where } c > 0 \text{ and the } b_i \text{ are arbitrary except that at least one, say } b_j, \text{ is not zero,} \tag{16}$$

or

$$a_i/b_i > 0 \quad (i = 1, \cdots, n). \tag{17}$$

Then given any x_0, z_0 the solution $x(t), z(t)$ of (15) *satisfying $x(0) = x_0$, $z(0) = z_0$ exists on $0 \leqq t < \infty$ and*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} z(t) = 0. \tag{18}$$

In (3) $-\infty < z < \infty$ means $-\infty < z_i < \infty$ $(i = 1, \cdots, n)$, and $|z|$ in (4) means $|z| = \sum |z_i|$.

The proof of Theorem 2 differs only in minor details from that of Theorem 1; appropriate energy functions analogous to (13) are defined for the case of (16) and (17). An obvious analogue of Corollary 1 holds here when $e_i(t) \to 0$ as $t \to \infty$.

Consideration of the linear special case of (15) defined by

$$h_i(t, x, z) \equiv h_i > 0, \quad f(x) = x, \; e_i(t) \equiv 0, \tag{19}$$

where the h_i are constants, sheds additional light on conditions (16, 17). It can be shown that the real part of each of the characteristic roots of the coefficient matrix associated with (15, 19) is negative if either (16) or (17) is satisfied. Hence, in the special case of (19), Theorem 2 reduces to a well-known theorem of Liapounov (see [1, p. 314]). This classical theorem may be used together with a standard perturbation technique to obtain a local (but not global) asymptotic stability result if, for example, the second condition in (19) is generalized to $f(x) = x + o(x)$ as $x \to o$. However, a perturbation technique is hopeless if, say, $f(x) = x^3$ (which satisfies (5, 6, 7)).

We remark that the last n equations of (15) may be written as

$$\dot{z} = G(t, x, z)z + b f(x) + e(t),$$

where

$$G = -\operatorname{diag}(h_1, \cdots, h_n), \quad z = \operatorname{col}(z_1, \cdots, z_n),$$
$$b = \operatorname{col}(b_1, \cdots, b_n), \quad e = \operatorname{col}(e_1, \cdots, e_n).$$

However, if G is of the form $G = -h(t, x, z)A$, where $h(t, x, z)$ is a scalar function satisfying (3, 4) and A is a positive definite real symmetric matrix, then the system is easily transformed into a special case of (15). A similar comment applies to (20) below. This observation is used in the reactor applications.

Theorem 3 is concerned with the system

$$\begin{aligned} \dot{x} &= -a_0 y - \sum_{i=1}^{n} a_i z_i \\ \dot{y} &= b_0 f(x) + e_0(t) \\ \dot{z}_i &= -h_i(t, x, y, z) z_i + b_i f(x) + e_i(t) \quad (i = 1, \cdots, n), \end{aligned} \tag{20}$$

where again the a_i and b_i are constants and $z = (z_1, \cdots, z_n)$.

THEOREM 3. *Let the functions h_i, f, e_i be sufficiently smooth for a local existence and uniqueness theorem to hold for* (20) *on $0 \leqq t < \infty$; $-\infty < x, y, z_i < \infty$. Let the h_i satisfy* (3, 4); *the f,* (5, 6, 7); *and the e_i,* (8, 9). *Furthermore, let the constants a_i, b_i satisfy either*

$$a_i = c\, b_i, \text{ where } c > 0 \text{ and the } b_i \text{ are arbitrary except that } b_0 \neq 0 \text{ and at least one other } b_i, \text{ say } b_j, \text{ is not zero,} \tag{21}$$

or

$$a_i/b_i > 0 \quad (i = 0, \cdots, n). \tag{22}$$

Then given any x_0, y_0, z_0 the solution $x(t)$, $y(t)$, $z(t)$ of (20) *satisfying $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$ exists on $0 \leqq t < \infty$ and*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} z(t) = 0. \tag{23}$$

Here the range of the variables in (3) is $0 \leqq t < \infty$; $-\infty < x, y, z_i < \infty$ and the condition in (4) is $0 \leqq t < \infty$; $|x|, |y|, |z| \leqq B$. An obvious analogue of Corollary 1 holds here when $e_i(t) \to 0$ as $t \to \infty$.

Similar to (19) we consider the special linear case of (20) defined by

$$h_i(t, x, y, z) \equiv h_i > 0, \quad f(x) = x, \quad e_i(t) \equiv 0, \tag{24}$$

where the h_i are constants. Here it can also be shown that the real part of each of the characteristic roots of the coefficient matrix associated with (20, 24) is negative if either (21) or (22) is satisfied. The remarks following (19) are equally applicable here.

On comparing Theorems 2 and 3 it might be suspected that more than one y type component could appear in (20) without causing any change in the conclusion of Theorem 3. Consideration of the example $\dot{x} = -y_1 - y_2 - z$, $\dot{y}_1 =$

$f(x)$, $\dot{y}_2 = f(x)$, $\dot{z} = -z + f(x)$, where f satisfies (5, 6, 7), shows that this is not the case.

In the proof of Theorem 3, under hypothesis (21), the energy function

$$E(x, y, z) = \frac{1}{c} g(x) + \frac{1}{2} y^2 + \frac{1}{2} \sum_{i=1}^{n} z_j^2 \tag{25}$$

is employed. The periodicity of all solutions of the autonomous system

$$\dot{x} = -c\, b_0 y, \quad \dot{y} = b_0 f(x),$$

note (6), also plays an important role in the proof.

Finally, Theorem 2 with $n = 1$ (which is really Theorem 1) and Theorem 3 are applied to certain problems in reactor dynamics. These enable us to consider not only more general reactor models than were considered previously in [5], but also to obtain global rather than local results. To illustrate the applicability of Theorem 3, consider the dynamic equations for a class of heterogeneous reactors of $m(m \geqq 2)$ media:

$$\begin{aligned} \dot{u} &= -\sum_{j=1}^{m} \alpha_j T_j \\ \epsilon_i \dot{T}_i &= -h(t, u, T) \sum_{j=1}^{m} X_{ij}(T_i - T_j) + \eta_i f(u) + e_i(t) \quad (i = 1, \cdots, m), \end{aligned} \tag{26}$$

where $T = \text{col}\,(T_1, \cdots, T_m)$. The special case studied in [5] had $h(t, u, T) \equiv 1$, $f(u) = \exp[u] - 1$, $e_i(t) \equiv 0$ $(i = 1, \cdots, m)$. In (26) u is the logarithm of the ratio of the reactor power to the stationary power ($u \equiv 0$ at equilibrium) and T_i is the deviation of temperature from equilibrium temperature in the ith medium ($T_i \equiv 0$ at equilibrium). The quantities ϵ_i, α_i, X_{ij}, η_i are the reactor parameters (for details see [5]), where $X_{ij} = X_{ji}$ and $\sum_{i=1}^{m} \eta_i = 1$, $\epsilon_i > 0$ for physical reasons; h is a heat conduction term and $e_i(t)$, $i = 1, \cdots, m$ are forcing terms. Letting $\alpha = \text{col}\,(\alpha_1, \cdots, \alpha_m)$, similarly η and e, and $\mu' =$ transpose of μ (for any μ), and $\epsilon = \text{diag}\,(\epsilon_1, \cdots, \epsilon_m)$, $A = (A_{ij})$, where $A_{ii} = \sum_{j \neq i} X_{ij}$, $A_{ij} = -X_{ij}$ $(i \neq j)$, (26) can be written as

$$\begin{aligned} \dot{u} &= -T'\alpha \\ \epsilon\dot{T} &= -h(t, u, T)AT + \eta f(u) + e(t). \end{aligned} \tag{27}$$

If $X_{ij} = X_{ji} > 0$ $(i \neq j)$, it can be shown that the real symmetric matrix A in (27) has precisely one characteristic value equal to zero and that the remaining $m - 1$ characteristic values are positive. Moreover, if $\epsilon_i > 0$, there exists a real nonsingular constant matrix R such that

$$R'\epsilon R = I, \quad R'AR = D = \text{diag}\,(d_1, \cdots, d_m), \tag{28}$$

where I is the m by m unit matrix. Furthermore, exactly one d_i is zero, say $d_j = 0$, and $d_i > 0$ $(i \neq j)$. Letting $T = RQ$ in (27) and

$$a = R'\alpha, \quad b = R'\eta, \quad \tilde{e}(t) = R'e(t), \tag{29}$$

(27) becomes

$$\begin{aligned} \dot{u} &= -a_j Q_j - \sum_{k \neq j} a_k Q_k \\ Q_j &= b_j f(u) + \tilde{e}_j(t) \qquad (30) \\ Q_i &= -h(t, u, RQ)\, d_i Q_i + b_i f(u) + \tilde{e}_i(t) \quad (i \neq j), \end{aligned}$$

which, under the above assumptions regarding ϵ_i and X_{ij}, has the same stability properties as (26), since R is a nonsingular constant matrix, and (30) obviously has the form of (20).

THEOREM 4. *Let h, f, e_i satisfy the hypothesis of Theorem 1 and be sufficiently smooth to guarantee local existence and uniqueness for (26); let $X_{ij} = X_{ji} > 0$, $\epsilon_i > 0$, $i = 1, \cdots, m$; let R be chosen as in (28); and let a, b be defined by (29). Further, let a, b satisfy either*

(31) *$a = c\, b$, where $c > 0$ and b is arbitrary except that $b_j \neq 0$ and at least one other b_i is not zero,*

or

(32) $$a_i/b_i > 0.$$

Then given any u_0, T_0 the solution $u(t)$, $T(t)$ of (26) for which $u(0) = u_0$, $T(0) = T_0$ exists for $0 \leqq t < \infty$ and $\lim_{t\to\infty} u(t) = 0$, $\lim_{t\to\infty} T(t) = 0$.

Note that the verification of (31) or (32) is a purely algebraic matter; (31) is analogous to the crucial assumption (2.6) of [6] used in the analysis of a certain continuous medium reactor.

Added in proof: The complete paper has appeared in Arch. for Rat. Mech. and Anal., 5 (1960), 194–211. For further generalizations see J. J. Levin, same journal, 6 (1960), 65–74.

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