CONVERSE THEOREMS OF LYAPUNOV'S SECOND METHOD

By J. L. Massera

1. Introduction

In a recent report by S. Lefschetz and J. P. LaSalle it is stated, concerning the question of converse theorems, that, "One virtually witnesses the end of an investigation." Although this type of forecast is usually dangerous, it is a fact that such a high degree of perfection has been attained in this field that one is strongly tempted to concur in the above mentioned opinion.

We shall try to give a short account of the present state of the problem. It should be mentioned that a very comprehensive book on this question has recently been published by W. Hahn [3]. On the other hand, we shall not consider the closely related problems of boundedness, ultimate boundedness, and the like, which have been successfully investigated by T. Yoshizawa as regards both direct and converse theorems. Neither shall we discuss problems on special types of asymptotic behavior of the solutions based on the order of smallness of the Lyapunov function and its derivative (cf. [6], [22]).

We shall use the following notation. The systems considered are of the form

$$\dot{x} = f(x, t);$$

x is an element of a vector space X (which is generally assumed to be Euclidean); t is a real variable which varies over $J = [0, \infty)$; f is defined in a domain $G \subset X \times J$ which contains the t-axis $\{0\} \times J$ in its interior; $f(0, t) \equiv 0$. Lyapunov's second method studies the stability properties of the solution $x = 0$ by means of certain properties of a "Lyapunov function" $V(x, t)$ and its "total derivative" or "derivative along the solutions of the system", $V'(x, t)$. If X is Euclidean n-space and V is assumed to be continuously differentiable, $V'(x, t) = (\partial V/\partial x) \cdot f(x, t) + (\partial V/\partial t)$, where $\partial V/\partial x$ represents the gradient of V with respect to x and the dot a scalar product; but V' may be constructively defined under much more general assumptions. It is always assumed that $V(0, t) \equiv 0$.

The theorems whose converse is sought are the following:

(1) (Simple stability [11]). *If a positive definite V exists such that $V' \leqq 0$, $x = 0$ is stable.*

(2) (Uniform stability [17]). *If a positive definite V exists which has an infinitely small upper bound and such that $V' \leqq 0$, $x = 0$ is uniformly stable.*

(3) (Uniform asymptotic stability [11], [12]). *If a positive definite V exists which has an infinitely small upper bound and such that V' is negative definite, $x = 0$ is uniformly asymptotically stable.*

(4) (Instability [11]). *If a Lyapunov function V exists which has an infinitely small upper bound, such that V' is positive definite and V assumes positive values at points arbitrarily near $x = 0$, then $x = 0$ is unstable.*

(5) (Instability [11]). *If a bounded function V exists such that $V' = \lambda V + W$,*

$\lambda > 0$, $W \geqq 0$ *and, if* $V > 0$ *at points arbitrarily near* $x = 0$, *then* $x = 0$ *is unstable.*

(6) (Instability [2]). *Assume that there is a function* V *and a domain* H *having points arbitrarily near* $x = 0$, *such that* V *is positive and bounded and* $V' > \varphi(V)$ *in* H, *where* $\varphi(r)$ *is a continuous increasing function of* r *for* $r > 0$, $\varphi(0) = 0$. *Then* $x = 0$ *is unstable.*

In assessing the strength of converse theorems, three aspects are important:

(a) That the theorems hold under the weakest possible regularity assumptions on f and provide Lyapunov functions which satisfy the strongest possible regularity requirements. We shall use the following notation: C represents the class of all continuous functions; C_0 the class of functions satisfying a local Lipschitz condition at each point; C_s, $0 < s < \infty$, the class of functions having continuous derivatives up to the order s; C_∞ the class of functions having continuous derivatives of all orders. A bar placed over the C denotes the uniform boundedness with respect to t of the functions, Lipschitz constants or derivatives involved; for instance, $\bar{C}_s$ is the class of functions having continuous derivatives up to the order s which are uniformly bounded with respect to t.

(b) That V be defined in the largest possible domain and not merely in a sufficiently small neighborhood of $x = 0$.

(c) That the theorems hold for equations defined in general spaces (Banach spaces, dynamical systems in metric spaces, etc.).

We shall briefly refer to problems (b) and (c) in Part V. As to problem (a), the most effective method of attack used so far (Kurzweil, Massera) has been to find first a Lyapunov function V_0 which satisfies the essential requirements of the theorems except that $V_0 \in C_0$ and then to smooth out V_0 to obtain $V \in C_\infty$. The smoothing-out operation consists basically in a convolution transform with a kernel belonging to C_∞.

In what follows we shall denote the converse theorems with the same number as the direct statements, primed; the statements are therefore omitted except for the regularity conditions.

2. Simple and uniform stability

(1′) (Persidskiĭ [17]). $f \in C_s$, $V \in C_s$.
(2′) (Krasovskiĭ [5]). $f \in \bar{C}_1$, $V \in \bar{C}_1$.
(2′) (Kurzweil [8]). $f \in C_1$, $V \in C_1$.
(1′) (Yoshizawa [19]). $f \in C$, V *continuous at* $\{0\} \times J$.

It should be stressed that the last result is in a sense deceptive, because even continuity of V for $x \neq 0$ is not assured. Now, the gist of Lyapunov's second method lies precisely in the possibility of deducing stability properties from properties of V and V', where the latter can be computed *directly, without a previous integration of the system*; but this direct computation requires that V be sufficiently regular (for a detailed discussion of this point, cf. [16]).

As a matter of fact, Kurzweil and Vrkoč [10] have shown by means of a simple counterexample that if $f \in C$ and uniqueness of the solutions fails to hold, there may not exist a continuous V satisfying the requirements of Theorems (1) or (2) even if $x = 0$ is uniformly stable. To obtain a converse theorem in the case $f \in C$ they have introduced a new concept of stability, which we conventionally denote by *-stability and which differs from the usual one, roughly speaking, in that the solution $x = 0$ is compared not only with perturbed *exact* solutions of the equation but also with (in some sense) *approximate* solutions. It is shown that, if uniqueness holds, both concepts coincide. With this concept of stability the following converse theorems hold:

(1*) (Kurzweil-Vrkoč, [10]). $f \in C$, $V \in C_\infty$.
(2*) (Kurzweil-Vrkoč, [10]). $f \in C$, $V \in C_\infty$.

As a consequence of the previous remark, (1*) and (2*) reduce to (1′) and (2′) if uniqueness of the solutions is assumed.

3. Asymptotic stability

The most conspicuous results are:

(3′) (Massera, [13]). $f \in C_1$ *periodic*, $V \in \bar{C}_1$.
(3′) (Barbašin, [1]). $f \in C_s$ *autonomous*, $V \in \bar{C}_s$, $s \geqq 1$.
(3′) (Malkin, [12]). $f \in \bar{C}_1$, $V \in \bar{C}_1$.
(3′) (Krasovskiĭ, [7]). $f \in \bar{C}_1$, $V \in \bar{C}_1$ (as a corollary of a much more general theorem).
(3′) (Massera, [15]). $f \in C_0$, $V \in C_\infty$.
(3′) (Kurzweil, [9]). $f \in C$, $V \in C_\infty$.

In the first of Massera's papers, as well as in Malkin's, $V(x, t)$ is expressed as the integral from t to ∞ of $G(x(\tau))$, where $x(\tau)$ is the solution through (x, t) and G is a suitable gauge function which ensures the convergence of the integral. In the second paper of Massera, as was explained in the Introduction, a function V_0 is constructed (which is, roughly speaking, a supremum of $G(x(\tau))$ for $\tau \geqq t$) and then V_0 is smoothed out to obtain V. Kurzweil combines these methods and other technical resources with his idea of approximate solutions; owing to the very weak assumption on f his proof is very long and complicated.

Barbašin's proof is based on an entirely different idea, the so-called "method of sections". Krasovskiĭ puts the problem in a more general setting which enables him to obtain simultaneously converse theorems on asymptotic stability and on instability. He defines what he calls noncritical and uniform behavior or, as we prefer to render it, *uniformly noncritical behavior*, as follows:

The solutions of a differential system have a uniformly noncritical behavior in the cylinder $\| x \| < \delta$, $t \geqq 0$ (which is supposed to be contained in G) if given η, $0 < \eta < \delta$, there is a $T = T(\eta) > 0$ such that the solution $x(t)$ passing through any (x_0, t_0), $\| x_0 \| > \eta$, $t_0 \geqq T$, satisfies $\| x(\tau) \| > \delta$ for some $\tau = \tau(x_0, t_0)$ in $(t_0 - T, t_0 + T)$. If we always have $\tau < t_0$ and if $f \in \bar{C}$, this behavior is equivalent to uniform asymptotic stability. If the system is autonomous, a necessary and sufficient condition for uniformly noncritical behavior is that the sphere $\| x \| < \delta$ contain no non-trivial complete trajectory (i.e., a trajectory lying in the sphere for $-\infty < t < +\infty$).

The following theorems can then be proved:

THEOREM A. (Krasovskiĭ, [4]). *If $f \in C_1$ autonomous, a necessary and sufficient condition for the existence of a $V \in \bar{C}_1$ with a definite V' is that there exist a neighborhood of $x = 0$ containing no complete trajectory except the trivial one.*

THEOREM B. (Krasovskiĭ, [7]). *If $f \in \bar{C}_1$, a necessary and sufficient condition for the existence of a $V \in \bar{C}_1$ with a definite V' is that the behavior be uniformly noncritical in some cylinder $\| x \| < \delta$, $t \geqq 0$.*

Krasovskiĭ's Theorem (3′) above is a corollary of Theorem B. The method of proof of Theorem A is a combination of the method of sections and integral representations of the type of Massera's. The method of proof of Theorem B is much more elementary.

4. Instability

Theorem (4), as it stands, admits no converse, because its assumptions imply much more than mere instability. On the other hand, Theorem (5) is a corollary of Theorem (6), so that any Theorem (5′) implies (6′).

(4*) (Krasovskiĭ, [7]). *A necessary and sufficient condition for the existence of a V satisfying the assumptions of Theorem (4) is that the behavior be uniformly noncritical and that there exist $t_0 \geqq 0$ and $x_n \to 0$ such that $\tau(x_n, t_0) > 0$.*

(5′) (Vrkoč, [18]). $f \in C_1$, $V \in C_1$, $V' = V$ ($\lambda = 0$, $W \equiv 0$).

5. Other refinements

We now briefly refer to problems (b) and (c) (cf. Introduction).

The question of defining V in the largest possible domain (which includes the case of asymptotic stability in the large) has been partly solved by Kurzweil and Zubov. Zubov proved in 1955 [20] the following result (with slight changes in the formulation):

(3′) *If $f \in C_s$ autonomous, $s \geqq 1$, a $V \in C$ exists in the domain $A \subset X$ of asymptotic stability (the attractive domain of $x = 0$), $V \to \infty$ as x approaches the boundary of A.*

The proof of Kurzweil's (3′) defines V in any domain $H \times J$ contained in the domain of asymptotic stability in $X \times J$; if the system is autonomous, this leads again to Zubov's result by taking $H = A$. It is not unlikely that, with small modifications in the proof, Kurzweil's method will yield the definition of V in the domain of asymptotic stability in the general non-autonomous case. Zubov has also tackled this general case, but he obtains only $V \in C$, which is very unsatisfactory (cf. previous remark to Theorem (1′) of Yoshizawa).

The possibility of extending the converse theorems to systems defined in spaces which are more general than finite-dimensional Euclidean is closely related to the problems of regularity. Indeed, in a general space there may not exist a convenient notion of differential (needed for the definition of V') and, even if it exists, there may not be "enough" smooth functions in the space. For instance, it is easy to show that in a real Banach space which is not isomorphic to Hilbert space there are no positive definite quadratic forms. Zubov [21] has proved converse theorems in the case of general dynamical systems but he obtains only $V \in C$, which is unavoidable in this general setting but deceptive in the sense explained in the remark made before. Another result in this general direction is:

(3′) (Massera, [14], [15]). *$f \in C_s$, $V \in C_s$, $0 \leqq s < \infty$ in any Banach space where gauge functions $\in C_s$ exist; in particular, in any Banach space whatsoever, $f \in C_0$, $V \in C_0$.*

Cf. also [16] and the report on Linear Differential Equations and Functional Analysis in this Symposium.

UNIVERSITY OF MONTEVIDEO

BIBLIOGRAPHY

[1] E. A. BARBAŠIN, *The method of sections in the theory of dynamical systems*, Mat. Sb. (N. S.), 29 (1951), 233–280 (Russian).

[2] N. G. ČETAEV, *Un théoreme sur l'instabilité*, C. R. (Doklady) Acad. Sci. URSS, 1934 I, 529–531.

[3] W. HAHN, *Theorie und Anwendung der direkten Methode von Ljapunov*, Erg. der Math., Heft 22, Berlin 1959.

[4] N. N. KRASOVSKIĬ, *On the converse of the theorems of A. M. Lyapunov and N. G. Četaev on instability for autonomous systems of differential equations*, Prikl. Mat. Meh., 18 (1954), 513–532 (Russian).

[5] N. N. KRASOVSKIĬ, *On the inversion of the Theorem of K. P. Persidskiĭ on uniform stability*, Prikl. Mat. Meh., 19 (1955), 273–278 (Russian).

[6] N. N. KRASOVSKIĬ, *On the inversion of the theorems of Lyapunov's second method and the problem of the stability of motion in the first approximation*, Prikl. Mat. Meh., 20 (1956), 255–265 (Russian).

[7] N. N. KRASOVSKIĬ, *On the theory of the second method of A. M. Lyapunov for the investigation of stability*, Mat. Sb. (N. S.), 40 (1956), 57–64 (Russian).

[8] J. KURZWEIL, *On the reversibility of the first theorem of Lyapunov concerning the stability of motion*, Čehosl. Mat. Žurnal, 5 (1955), 382–398 (Russian; English summary).

[9] J. KURZWEIL, *On the converse of the second theorem of Lyapunov on the stability of motion*, Čehosl. Mat. Žurnal, 6 (1956), 217–259; 455–484 (Russian; English summary).

[10] J. KURZWEIL and I. VRKOČ, *The converse theorems of Lyapunov and Persidskij concerning the stability of motion*, Čehosl. Mat. Žurnal, 7 (1957), 254–274 (Russian; English summary).

[11] A. M. LIAPOUNOFF, *Problème général de la stabilité du mouvement*, Annals of Math. Studies No. 17, Princeton 1947.

[12] I. G. MALKIN, *On the question of the converse of Lyapunov's theorem on asymptotic stability*, Prikl. Mat. Meh., 18 (1954), 129–138 (Russian).

[13] J. L. MASSERA, *On Lyapunov's conditions of stability*, Annals of Math., 50 (1949), 705–721.

[14] J. L. MASSERA, *Sobre la estabilidad en espacios de dimensión infinita*, Revista Unión Mat. Argentina, 17 (1955), 135–147.

[15] J. L. MASSERA, *Contributions to stability theory*, Annals of Math., 64 (1956), 182–206.

[16] J. L. MASSERA and J. J. SCHÄFFER, *Linear differential equations and functional analysis. III. Lyapunov's second method in the case of conditional stability*, Annals of Math., 69 (1959), 535–574.

[17] K. P. PERSIDSKIĬ, *Ueber einen Satz von Ljapunov*, C. R. (Doklady) Acad. Sci. URSS, 14 (1937), 541–544.

[18] I. VRKOČ, *On the inverse theorem of Četaev*, Čehosl. Mat. Žurnal, 5 (1955), 451–461 (Russian; English summary).

[19] T. YOSHIZAWA, *On the stability of solutions of a system of differential equations*, Memoirs Coll. Sci. Univ. Kyoto, Ser. A, 29 (1955), 27–33.

[20] V. I. ZUBOV, *Problems of the theory of Lyapunov's second method, construction of the general solution in the domain of asymptotic stability*, Prikl. Mat. Meh., 19 (1955), 179–210 (Russian); cf. also [23].

[21] V. I. ZUBOV, *On the theory of A. M. Lyapunov's second method*, Doklady Akad. Nauk SSSR, 100 (1955), 857–859 (Russian); cf. also [23].

[22] V. I. ZUBOV, *Conditions for asymptolic stabiliiy in the case of non-stationary motion and estimate of the rate of decrease of the general solution*, Vestnik Leningradskogo Univ., 12 (1957), 110–129; 208 (Russian); cf. also [23].

[23] V. I. ZUBOV, *Methods of A. M. Lyapunov and their application*, Izd. Leningradskogo Univ., Leningrad 1957 (Russian).