ON THE INTEGRABILITY OF AREA PRESERVING CREMONA MAPPINGS NEAR AN ELLIPTIC FIXED POINT

By J. Moser

Section 1

In recent times there appeared several papers concerning the question of divergence of the series in perturbation theory of classical mechanics or the related question of nonexistence of integrals of Hamiltonian systems (see [4], [6], [7], [10]). The aim of this paper is to give a very simple model of Hamiltonian systems for which the divergence of the series involved can be exemplified very easily with algebraic means.

The simplest problem of this type was investigated by G. D. Birkhoff [1]: Consider a mapping

$$\begin{aligned} x_1 &= f(x, y) \\ y_1 &= g(x, y) \end{aligned} \tag{1}$$

near a fixed point $x = y = 0$, assuming that $f(x, y)$, $g(x, y)$ are real analytic near the origin. Furthermore, the mapping is assumed to be area-preserving, a groperty which is related to Hamiltonian systems of differential equations. The fixed point $x = y = 0$ of (1) is called elliptic if the linear part of the mapping is equivalent to a rotation about an angle α. Therefore we assume

$$f = x \cos \alpha - y \sin \alpha + f_2 + f_3 + \cdots$$

$$g = x \sin \alpha + y \cos \alpha + g_2 + g_3 + \cdots$$

where f_ν , g_ν represent homogeneous polynomials of degree ν in x, y.

Birkhoff's result can be formulated as follows: If α/π is irrational, then there exist real formal series $\varphi(\xi, \eta)$, $\psi(\xi, \eta)$ such that the formal transformation

$$\begin{aligned} x &= \varphi(\xi, \eta) = \xi + \cdots \\ y &= \psi(\xi, \eta) = \eta + \cdots \end{aligned} \tag{2}$$

transforms the mapping (1) into

$$\begin{aligned} \xi_1 &= \xi \cos \omega - \eta \sin \omega \\ \eta_1 &= \xi \sin \omega + \eta \cos \omega \end{aligned} \tag{3}$$

where

$$\omega = \alpha + \beta(\xi^2 + \eta^2) + \cdots$$

is a formal real power series in $\xi^2 + \eta^2$. In other words, in appropriate coordinates the mapping (1) has the form of a rotation where the angle of rotation ω depends on the radius; however, the coordinate transformation (2) might be given by divergent series.

If φ, ψ are convergent, hence also ω convergent, it is clear that the function $\xi^2 + \eta^2$ is invariant under the mapping (3), hence there is an analytic function $G(x, y) = x^2 + y^2 + \cdots$ such that $G(x_1, y_1) = G(x, y)$. In fact, Birkhoff showed that the existence of such an invariant function $G(x, y)$ is necessary and sufficient for the convergence of φ, ψ in (2) with appropriate normalizations. Since such an invariant function $G(x, y)$ corresponds to an integral of a system of differential equations, Birkhoff called mappings (1), for which (2) converges integrable, the others nonintegrable.

The question arises which case is the "general" one. Since the divergence of the series φ, ψ is due in part to the presence of small divisors of the form $\lambda^n - 1 (\lambda = e^{i\alpha})$, it was conjectured that one might have convergence if α/π is a number which is badly approximated by rational numbers, i.e., which satisfy, say,

$$\left| \frac{\alpha}{\pi} - \frac{p}{q} \right| \geqq cq^{-3} \tag{4}$$

for all integers p, q with some positive c. For discussion in this direction see Wintner [3], Petersson [2]. It is well known that the numbers α which do not satisfy (4) are of measure zero. Such considerations would indicate that the divergence of (2) is exceptional.

On the other hand, it was shown quite recently that the integrable case is the exceptional one, although in the sense of Baire's categories and not in the sense of measure theory. This was carried out for the mappings of the form (1) by Rüssmann [10], using the same ideas and methods developed by C. L. Siegel [4] for Hamiltonian systems near an equilibrium.

Here we want to give explicit examples of mappings for which the non-integrability can be established very easily. Moreover, these examples show very clearly that number-theoretical conditions like (4) are completely irrelevant. For example, we will show that the mapping

$$\begin{aligned} x_1 &= (x + y^3) \cos \alpha - y \sin \alpha, \qquad \alpha/\pi \text{ not integer}, \\ y_1 &= (x + y^3) \sin \alpha + y \cos \alpha, \end{aligned} \tag{5}$$

is always nonintegrable. Moreover, we establish simple conditions for entire Cremona transformations which guarantee nonintegrability. Even though the arguments given in this note prove only the divergence of the transformation (2) into normal form and hence exclude real analytic integrals of the form $G(x, y) = x^2 + y^2 + \cdots$ it is easily seen that even differentiable integrals are excluded. This follows from Lemma 1 in [6].

The proofs of the nonexistence of integrals proposed by Diliberto [7] are of different nature, since these integrals are considered in the neighborhood of a periodic surface, while the problem discussed here refers to integrals near a periodic solution.

Section 2

To construct such area preserving nonintegrable mappings (1) we study entire Cremona transformations, i.e., mappings for which $f(x, y)$, $g(x, y)$ are polynomials. From the fact that

$$f_y g_x - f_x g_y = 1$$

it follows that the inverse mapping is also given by polynomials:

$$x = \hat{f}(x_1, y_1)$$
$$(6) \qquad y = \hat{g}(x_1, y_1)$$

(see W. Engel [8]).

Let k denote the larger of the degrees of f and g; similarly, $\hat{k}$ the larger of the degrees of $\hat{f}$, $\hat{g}$. Let F, G be the homogeneous polynomials of degree k which are contained in f, g respectively. Since $f_y g_x - f_x g_y = 1$ it follows that $F_y G_x - F_x G_y = 0$ if $k > 1$ which implies that

$$aF + bG = 0; \qquad a^2 + b^2 = 1 ;$$

without loss of generality we can assume

$$(7) \qquad G = \mu F.$$

Similarly for the inverse

$$\hat{G} = \hat{\mu}\hat{F}.$$

The number μ has a geometrical interpretation for the mapping: For large $x^2 + y^2$ the image point will lie near the line $y_1 = \mu x_1$.

Lemma. *If the Cremona transformation* (1), *denoted by* M, *satisfies* $k = \hat{k} > 1$ *and*

$$(\mu - \hat{\mu})F(1, \mu)\hat{F}(1, \hat{\mu}) \neq 0$$

and if $F(x, y)$, $\hat{F}(x, y)$ *are relatively prime over the ring of real polynomials, then the iterates* M^{2q} *of even order have only finitely many fixed points*; *in fact, their number is at most* k^{2q}.

Proof. For all integers ν we denote by (x_ν, y_ν) the coordinates of the image point of x, y under M^ν. The fixed points of M^{2q} satisfy

$$x_{2q} = x_0 ; \qquad y_{2q} = y_0 ,$$

which are in one to one correspondence with the solutions of

$$x_q - x_{-q} = 0$$
$$(8) \qquad y_q - y_{-q} = 0.$$

x_q, x_{-q} represent polynomials in x, y for which we easily compute the highest order terms by induction:

$$x_q = F^{(k^q-1)}(x, y)a_q + \cdots$$
$$y_q = \mu F^{(k^q-1)}(x, y)a_q + \cdots$$

where

$$a_q = F(1, \mu)^{(k^{q-1}-1)/k-1}.$$

There are similar equations for x_{-q}, y_{-q}. Thus the left hand sides of (8) represent two polynomials of degree k^q.

According to a well known theorem of Bezout the number of common root exceeds the product of the degrees of these polynomials—i.e., k^{2q}—only if the polynomials have a common factor. Thus the Lemma will be proven if it is shown that the polynomials $x_q - x_{-q}$ and $y_q - y_{-q}$ have no common factor, for which it suffices to show that their principal parts have no common factor. These principal parts are

$$a_q F^{k^{q-1}} - \hat{a}_q \hat{F}^{k^{q-1}}$$

$$\mu a_q F^{k^{q-1}} - \hat{\mu} \hat{a}_q \hat{F}^{k^{q-1}}.$$

They have a common factor only if

$$a_q F^{k^{q-1}} \quad \text{and} \quad \hat{a}_q \hat{F}^{k^{q-1}}$$

have a common factor, since $\mu \neq \hat{\mu}$. Since $a_q \hat{a}_q \neq 0$ this implies that F and $\hat{F}$ have a common factor which contradicts the assumption.

Section 3

To prove that a mapping M is nonintegrable it is sufficient to show that there is sequence of integers $q_1, q_2, \cdots q_\nu \to \infty$, such that M^{q_ν} has finitely many[1] fixed points in

$$0 < x^2 + y^2 \leqq \rho_\nu^2$$

where $\rho_\nu \to 0$. If, namely, (2) is convergent then one can assume the mapping to be of the form (3). Then with each fixed point P the whole circle through P consists of fixed points; i.e., their number is infinite.

To establish infinitely many fixed points of the iterates of M we use Birkhoff's fixed point theorem for which we refer to [5]. If

$$\frac{\alpha}{2\pi} \neq \frac{p}{q}; \qquad q = 1, 2, 3, 4$$

one can find a convergent transformation (2) such that M has the form (3) up to terms of order 3 inclusively. Under the additional assumption $\beta \neq 0$ Birkhoff's fixed point theorem states: For any $\epsilon > 0$ there is a q such that M^q has at least $2q$ fixed points in $x^2 + y^2 < \epsilon^2$. It is easily seen that one can restrict q to even numbers.

Section 4

Summarizing, we have shown that under the conditions of the Lemma and the additional assumption that (1) has an elliptic fixed point with $\beta \neq 0$ the

[1] I.e., at least one and not infinitely many fixed points.

mapping is nonintegrable near that fixed point. For the example (5) one easily checks that the conditions of the Lemma are satisfied, as well as $\beta = \frac{3}{8} \neq 0$ provided

$$\sin \alpha \neq 0.^{2}$$

The construction of Cremona transformations is achieved easily by repeated application of mappings

$$x_1 = x + h(y)$$
$$y_1 = y$$

and rotations. Since the conditions of the Lemma and $\beta \neq 0$ can be formulated by finitely many inequalities one finds that in the class of area-preserving Cremona transformation the nonintegrable case is the general situation.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE

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[2] The conditions $\alpha/6\pi$, $\alpha/8\pi$ is not an integer, are redundant in this case.