ON THE EXISTENCE OF STRONGLY RECURRENT AND PERIODIC ORBITS*

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Let M^n be a compact manifold of class C^2 and let V be a C^1 contravariant vector field defined on M^n. As is well known, the differential equations defined by V permit us to define a flow; i.e., the action of R^1 as a topological transformation group on M^n. The orbit of a point under the flow is a solution curve to the differential equations defined by the vector field.

Let us begin by recalling a number of standard definitions [Oxtoby; "Ergodic Sets", BAMS 1952].

Definition. A point p on M^n is called *quasi-regular* provided

$$\lim_{T\to\infty} (1/T) \int_0^T f(pt)\, dt$$

exists for every continuous function $f(x)$ defined on M^n.

From now on the Banach space consisting of all the continuous functions on M^n with the usual norm will be denoted by $C(M)$. For any quasi-regular point p, there is a unique finite invariant measure μ_p such that

$$\lim_{T\to\infty} (1/T) \int_0^T f(pt)\, dt = \int_M f(x)\, d\mu_p(x).$$

Definition. A quasi-regular point p is called *regular* provided
(1) $\mu_p(U) > 0$ for every neighborhood U of p.
(2) The flow on M is metrically transitive with respect to μ_p.

It is obvious that every regular point is strongly recurrent; i.e., has no neighborhood to which it returns with probability zero. We recall also the well-known fact that the regular points have measure 1 with respect to any normalized finite invariant measure.

Next let ω be any n-form of odd kind and class C^1 on the n-dimensional manifold M which is different from zero at every point of M, and defines a positive measure on M. Note that any other form $\bar{\omega}$ satisfying the conditions we have imposed on ω can be written $\bar{\omega} = e^g\omega$ where $g(x)$ is a C^1 function. We establish the convention that if α is any function or form on M, then α' is the Lie derivative of α with respect to our flow.

There is a unique function $f(p)$ such that $\omega' = f\omega$. This function will be denoted by $\mathrm{Div}_\omega\, p$. Given any point p of M, there is a coordinate neighborhood of p such that the measure defined by ω in this neighborhood coincides with euclidean measure. The ordinary divergence of V in such a neighborhood is the same as $\mathrm{Div}_\omega\, p$. In particular, at any point p where the vector field vanishes, $\mathrm{Div}_\omega\, p$ is the sum of the diagonal terms in the matrix of the linear approxima-

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tion to V at p, and is therefore independent of ω. This is a special case of a more general fact. For any quasi-regular point p, let

$$D(p) = \lim_{T\to\infty} (1/T) \int_0^T \mathrm{Div}_\omega\,(pt)\,dt.$$

LEMMA. *$D(p)$ is independent of ω.*

In fact, if $\bar{\omega} = e^g\omega$, then $\mathrm{Div}_{\bar{\omega}}\,p = \bar{\omega}'/\bar{\omega} = g'e^g\omega + e^g\omega'/e^g\omega = g' + \mathrm{Div}_\omega\,p$. But obviously two functions which differ by the Lie derivative of a function have the same average value on any quasi-regular orbit.

For points p which are stationary under the flow, $D(p)$ is the sum of the eigenvalues of the vector field at p, and for periodic points $D(P)$ is essentially the Poincaré index of the periodic orbit.

We will prove the following result:

THEOREM. *There is at least one regular point p for which $D(p) \geqq 0$ and at least one regular point q for which $D(q) \leqq 0$.*

Before proceeding to the proof, let us note some immediate consequences of the theorem.

COROLLARY. *If the sum of the eigenvalues of V is positive at each point where V vanishes, there exist strongly recurrent points which are not stationary.*

COROLLARY. *If M is the two-sphere and the sum of the eigenvalues of V is positive at each point where V vanishes, there exists a nonconstant periodic solution to the differential equations.*

All that is needed is the knowledge that regular points are strongly recurrent and that, on the surface of the two-sphere, the only such points are periodic.

We proceed to the proof. Since $\omega' = \lim_{\Delta\tau\to 0} \omega(p\Delta\tau) - \omega(p)/\Delta\tau$ it is clear that $\int_M (\mathrm{Div}_\omega p)\omega = \int_M \omega' = 0$. If we use P to denote the cone of strictly positive functions in the Banach space $C(M)$ this implies that $\mathrm{Div}_\omega p$ does not belong to P or $-P$, since ω defines a positive measure. Letting $\bar{\omega} = e^g\omega$ and using the fact that $\mathrm{Div}_{\bar{\omega}}p = g' + \mathrm{Div}_\omega p$, we see that in fact, for any C^1 function g, $g' + \mathrm{Div}_\omega p$ does not belong to P or to $-P$. From this it is clear that P contains no function of the form $g' + \lambda\,\mathrm{Div}_\omega p$, where λ is a scalar. In fact, if $\lambda \neq 0$, this follows by scalar multiplication, while if $\lambda = 0$ it follows by looking at any point where g assumes a maximum.

Let L be the set of all functions of the type $g' + \lambda\,\mathrm{Div}_\omega p$. Then L is a linear subspace of $C(M)$ such that $L \cap P = \emptyset$. Therefore, it is possible to define a positive linear functional on $C(M)$ which vanishes on L. Such a linear functional is representable by a non-negative measure μ defined on the Borel sets of M. Since by definition of μ, $\int_M g'\,d\mu = 0$ for every C^1 function g, the measure μ is invariant under the flow. Moreover, by definition of μ, $\int_M \mathrm{Div}_\omega p\,d\mu = 0$. Thus we have proved the following result:

LEMMA. *There exists a non-negative invariant measure* μ *such that*

$$\int_M \mathrm{Div}_\omega p \, d\mu = 0.$$

Finally, we recall that the set R of regular points comprises almost all of the points of M relative to the measure μ, and that the time average of $\mathrm{Div}_\omega p$ on the orbit through any regular point p is what we have denoted by $D(P)$. Therefore, $0 = \int_M \mathrm{Div}_\omega p \, d\mu = \int_R \mathrm{Div}_\omega p \, d\mu = \int_R D(p) \, d\mu(p)$. From this our theorem follows.

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