PERIODIC ORBITS ON TWO MANIFOLDS*

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Given any vector field V on an oriented two manifold, we can assign to any curve C an integer $I(V, C)$ defined by an integral. On the other hand, to any C with everywhere nonzero tangent vector, we can assign an integer called the winding number of C. By comparing these two integers, necessary conditions for C to be a periodic orbit may be obtained. These ideas are discussed in the first two sections, while in the third we apply them to a problem on the solid torus; we show that a certain class of vector field always admits a periodic orbit.

1. Winding number

Let M be a compact oriented two dimensional manifold of genus g; that is, a sphere with g handles. A regular curve on M is a curve of class C^1 with nowhere vanishing tangent vector. We have defined [2] the winding number by the formula

$$w(c) \equiv \frac{1}{2\pi} \int_{t=0}^{1} d(\dot{C}(t) - F(t)) \mod \chi(M)$$

in which:

C is a regular closed curve; that is $C(0) = C(1)$ and $\dot{C}(0) = \dot{C}(1)$.

$\dot{C}(t)$ is the tangent vector at parameter t.

F is a vector field on M, fixed once for all, which we shall specify precisely later.

$\dot{C}$-F is the angle from F to $\dot{C}$.

$\chi(M)$ is the Euler number of M, $\chi(M) = 2 - 2g$.

It is well known that the fundamental group of M may be represented by $2g$ generators $A_1, \cdots, A_{2g}$, with one relation $A_1A_2A_1^{-1}A_2^{-1} \cdots A_{2g-1}A_{2g}A_{2g-1}^{-1}A_{2g}^{-1} = 1$. Each of these generators may be represented by a regular simple closed curve through the base point. The vector field F is specified by the conditions that

(i) F has only one singular point.

(ii) The winding number of each of the curves A_i is zero.

Actually these conditions do not specify F uniquely, but they do determine the winding number uniquely.

Let us consider the case $g = 3$. A suitable field F is illustrated in Figure 1, which shows one half of M. Similarly labelled points are joined along the hidden side; thus there are five curves which tend at each end to the singular point. The remainder of M is filled by simple closed curves. Let us compute a few winding

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numbers. First of all, each of the interior boundary curves on the diagram has winding number zero, for each is never tangent to F. The outer boundary curve, on the other hand, has winding number 2 (mod 4), since along this curve the angle between its tangent and the field F makes two full rotations. More generally, from a similar diagram for genus g, we see that the outer boundary curve has winding number $g - 1$. In our paper [2], we have given a formula for computing the winding number of any simple closed curve from its homotopy class; we hope to prove in a later paper that this formula reduces to the statement that the absolute value of the winding number of a simple closed curve is equal to one less than the number of handles which it wraps around.

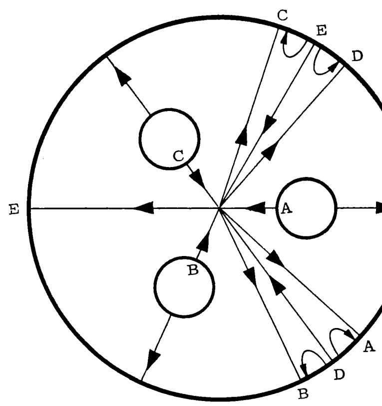

FIG. 1

2. Periodic orbits

Let V be a vector field on M having unique trajectories through nonsingular points. Then any periodic orbit C is a regular simple closed curve, so possesses a winding number. This winding number is computed by the formula

$$w(C) \equiv \frac{1}{2\pi} \int_C d(V - F) \quad \mathrm{mod}\, \chi$$

On the other hand, this latter formula is defined for any closed curve C, whether or not it is an orbit of V; if its value is not that prescribed for simple closed curves homotopic to C, then C cannot be an orbit. Following these ideas, we have proved [2]:

PROPOSITION. *Let S be the locus of singular points of V, C any closed curve not passing through any singular point, and w_0 the winding number of any simple closed curve homotopic to C. Suppose*

$$I(V, C) \equiv \frac{1}{2\pi} \int_C d(V - F) \not\equiv w_0 \quad \mathrm{mod}\, \chi$$

Then C is not homotopic on M-S to any orbit of V.

Now $I(V, C)$ is unchanged if V is replaced by any other field V' having the same singular set S and being homotopic to V. Let us apply this idea to F itself. Since $I(F, C) = 0$ for any C, the same is true for any F' homotopic to F. On the other hand, if C is the outer boundary of Figure 1, $w(C) = 2$. Hence, C is not an orbit of F'. This illustrates the general method for applying the winding number to questions of existence of periodic orbits.

3. The solid torus

Let M be a solid torus; that is, the Cartesian product of the closed two disc E^2 with the circle S^1. M is a three manifold with boundary B, where B is the product of two circles. A field W of oriented two planes on M is the assignment at each point of M of an oriented two dimensional subspace of the tangent space at that point; we assume that this assignment is made smoothly. If V is a vector field on M, we say that V is tangent to W if at each point P the vector $V(P)$ lies in the plane $W(P)$.

THEOREM. *Let W be a field of two planes on the solid torus such that $W(P)$ is the tangent plane to B whenever P is a point of B. Let V be a non-singular vector field tangent to W. Then V has a periodic orbit lying on B.*

PROOF. By the field of parallels on B, we mean the unit tangent vectors to the curves $P \times S^1$, where P is a boundary point of E^2. Any nonsingular vector field on B which admits no periodic orbit is homotopic to the field of parallels [1]. Also, giving a field of oriented planes W is equivalent to giving a nonsingular vector field orthogonal to it. By a field of 2 frames we mean the assignment at each point of an ordered pair of orthogonal unit vectors, the assignment being made smoothly over M. The above remarks show that our theorem is equivalent to proving that the field of two frames on B consisting of the parallels as first vectors, and the interior normals as second vectors, is not extendable to a field of two frames on M. Now the field of parallels is certainly extendable to a vector field on M; by obstruction theory [3], the homotopy classes of such extensions correspond one-one to the elements of $H^2(M, B; \pi_2(S^2))$ which is free cyclic. Representatives of each class may be constructed as follows: Suppose E^2 is a disc of radius 1, parametrized by polar coordinates (r, θ). Let ϕ be an angular coordinate on the circle S^1. Then the components of the vector field V_n are $\sin n(1 - r)\pi$ radially, $\cos n(1 - r)\pi$ in the direction of increasing ϕ, and 0 in the direction of θ. As $n = 0, \pm 1, \cdots$ the fields V_n represent each homotopy class exactly once. The set of all unit vectors orthogonal to V_n forms a circle bundle U_n over M, and the obstruction to extending the field of interior normals to a section of this bundle is an element of $H^2(M, B; \pi_1(S^1))$ which is also free cyclic. The extension is possible if and only if this obstruction is zero. Hence, our theorem is proved by the following lemma.

LEMMA. *The obstruction to sectioning U_n is a generator of $H^2(M, B; \pi_1(S^1))$.*

PROOF. It is sufficient to consider a typical disc $E^2 \times Q$. Any bundle over a disc is a product. To describe the product structure, we may give a fixed orthogonal homeomorphism between the fibre at the origin $r = 0$ and the fibre at each other point; such a mapping is determined by giving the images of a pair of orthogonal unit vectors. Consider the points lying on a single radius R_0. On any radius, V_n is tangent to the surfaces on which θ is constant. Hence, we define the desired mapping by sending the unit normal to this surface at the origin into the unit normal at P on R_0, and sending the vector in the surface making an angle $\pi/2$ with $V_n(0)$ into the similarly defined vector at $V_n(P)$. To compute the obstruction, we use this mapping to pull the vector at each boundary point back to the origin, thereby defining a mapping of the boundary into the fibre. It is easily seen that this map is a homeomorphism, hence generates $\pi_1(S^1)$. This completes the proof of the lemma.

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