STRUCTURAL STABILITY ON TWO-DIMENSIONAL MANIFOLDS

By M. M. Peixoto

1. Structural Stability

In this paper we give an abstract of the results we have obtained so far about structural stability on compact two-manifolds M^2; proofs will be given elsewhere.

Assume there is defined in M^2 a differentiable structure of class C^2 and also a Riemannian metric. We consider the set $\mathfrak{B}$ of all vector fields of class C^1 defined in M^2 and put in it the C^1-topology. There is no natural metric in $\mathfrak{B}$ but one arises naturally once we choose a covering of M^2 by a finite number of compact sets each contained in a coordinate neighborhood. A vector field $X \in \mathfrak{B}$ is said to be *structurally stable* when given any $\epsilon > 0$ we may find a neighborhood U of X such that whenever $Y \in U$ there is an ϵ-homeomorphism T (i.e., moving each point by less than ϵ) of M^2 onto itself mapping trajectories of X onto trajectories of Y. This important concept was introduced in 1937 by Andronov and Pontrjagin [1] for the case of the disc, assuming that the vector field is always without contact with the boundary. They gave then a characterization of structural stability and it is immediate that it is also valid for the case of the sphere S^2. Also valid for S^2 are the results of [2] that we established for the disc.

Our first theorem concerns the extension of the above characterization to the manifold M^2 under consideration.

2. The Characterization

Let X be a vector field of class C^1 defined in M^2. We have the following

Theorem 1. *A necessary and sufficient condition in order that X be structurally stable is that*

(1) *X has only simple singularities, the real part of the characteristic roots being different from zero;*

(2) *X has only a finite number of closed orbits and the stability index (sum of the characteristic exponents) of any of them is different from zero;*

(3) *no trajectory goes from saddle point to saddle point;*

(4) *the α- and ω-limit sets of any trajectory are either a singular point or a closed orbit.*

Conditions (1)–(3) are exactly the ones that are valid for the sphere S^2 so that to obtain a characterization for a general M^2 we need only to add the requirement (4) that excludes a complicated behaviour for the limit sets.

The proof that structural stability implies conditions (1)–(4) can be done without using the fact that the homeomorphism is an ϵ-homeomorphism; any homeomorphism would be enough. In other words, conditions (1)–(4) would still be a consequence of the following non-ϵ-definition of structural stability: X is structurally stable if there is a neighborhood U of X such that for any

$Y \in U$ there is a homeomorphism of M^2 onto itself transforming trajectories of X onto trajectories of Y. Therefore we have

THEOREM 2. *The ϵ and non-ϵ definitions of structural stability are equivalent.*

3. The Set Σ of all Structurally Stable Systems.

It is now clear that Σ is an open set of $\mathcal{B}$, a fact that follows immediately from the non-ϵ-definition but which requires Theorem 1 if we want to derive it from the ϵ-definition.

4. The Connected Components of Σ

Let us introduce an equivalence relation in $\mathcal{B}$ by calling equivalent two vector fields X, Y when there exists a homeomorphism of M^2 onto itself which is isotopic to the identity and such that it transforms trajectories of X onto trajectories of Y. Then $\mathcal{B}$ becomes divided into equivalence classes E_i. The meaning of the connected components of Σ is then clarified by the following

THEOREM 3. *There are only denumerably many indices i for which the interior $\mathring{E}_i$ of E_i is nonempty; $\mathring{E}_i$ coincides then with one connected component of Σ and conversely every such component is the interior of some E_i.*

5. The Density Theorem

We proved in [2] that, in the case of the sphere S^2, Σ is dense in $\mathcal{B}$, and it is very likely so in the case of any M^2. Actually Theorem 2 is a step in that direction. Beyond that we only proved that in the case of an orientable M^2 any vector field can be approximated by a vector field with only a finite number of closed orbits. Since Σ is open, if we can prove that it is also dense in $\mathcal{B}$ this means that "almost all" vector fields are structurally stable.

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INSTITUTO DE MATEMÁTICA PURA E APLICADA, RIO DE JANEIRO

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