PERIODIC SOLUTIONS AND INVARIANT SETS OF STRUCTURALLY STABLE DIFFERENTIAL SYSTEMS

By Lawrence Markus

Let M^n, $n \geqq 2$, be a compact differentiable (C^∞) manifold and let $\mathcal{B}$ be the Banach space of all C^1 contravariant vector fields, or first order differential systems, on M^n. Here we use the C^1-norm on a vector field

$$\mathcal{S}: \dot{x}^i = f^i(x^1, x^2, \cdots, x^n) \qquad i = 1, 2, \cdots, n.$$

That is,

$$\| \mathcal{S} \|_1 = \max \| f^i(P) \| + \max \| \operatorname{grad} f^i(P) \|$$

for $P \in M^n$, where we utilize an auxiliary Riemann metric on M^n and the topology of $\mathcal{B}$ is independent of the choice of this metric.

Two differential systems $\mathcal{S}$ and $\mathcal{S}'$ of $\mathcal{B}$ are called ϵ-homeomorphic in case there exists a homeomorphism Φ of M^n onto itself such that

(a) Φ carries the (sensed but not parametrized) solution curves of $\mathcal{S}$ onto those of $\mathcal{S}'$ and, vice versa, Φ^{-1} carries the solutions of $\mathcal{S}'$ onto those of $\mathcal{S}$.

(b) Φ moves each point of M^n a distance less than $\epsilon > 0$.

Definition. *A differential system $\mathcal{S} \in \mathcal{B}$ is structurally stable on M^n in case: for each $\epsilon > 0$ there exists $\delta > 0$ such that $\mathcal{S}' \in \mathcal{B}$, and $\| \mathcal{S} - \mathcal{S}' \|_1 < \delta$ implies that $\mathcal{S}'$ and $\mathcal{S}$ are ϵ-homeomorphic.*

We shall show that the critical points and the periodic solutions of a structurally stable differential system $\mathcal{S}$ on a compact manifold M^n are isolated and elementary. From a study of the minimal (invariant) sets we shall prove the existence of periodic solutions of $\mathcal{S}$. Thus, for structurally stable differential systems, the problem of Seifert [11] (the existence of a periodic solution of a non-critical differential system on the 3-sphere) and the Poincaré-Bendixson analogue (the existence of a periodic solution of a non-critical differential system penetrating a solid anchor ring) are resolved.

The theory of structurally stable differential systems in an open submanifold $\mathcal{O} \subset M^n$, with compact closure $\bar{\mathcal{O}}$ and smooth boundary $\partial\mathcal{O}$, is developed in a forthcoming paper [8].

Theorem 1. *Let $\mathcal{S}$ be a structurally stable differential system on M^n. Then each critical point of $\mathcal{S}$ is isolated, elementary, and topologically linear.*

Sketch of Proof. Let P be a critical point of $\mathcal{S}$ and, in local coordinates (x) centered at P, write

$$\mathcal{S}: \dot{x}^i = f^i(x^1, x^2, \cdots, x^n) \qquad i = 1, 2, \cdots, n$$

with $f^i(0) = 0$. Now approximate $\mathcal{S}$, in the C^1-norm, by $\mathcal{S}' \in \mathcal{B}$ on M^n such that near P we have

$$\mathcal{S}': \dot{x}^i = P^i(x^1, x^2, \cdots, x^n) \qquad i = 1, 2, \cdots, n$$

where the $P^i(x)$ are real polynomials. By choosing the coefficients of the polynomials $P^i(x)$ in general position we can require that every critical point of $\mathcal{S}'$ near P is isolated. Thus every critical point of $\mathcal{S}$ near P is isolated and hence P is an isolated critical point of $\mathcal{S}$, [9].

Now the eigenvalues of $\mathcal{S}$ at P are those of the matrix $(\partial f^i/\partial x^j)(0) = a^i_j$. It is easy to see that $\mathcal{S}$ is topologically linear at P (that is, there is a neighborhood N of P wherein $\mathcal{S}$ is homeomorphic to $\dot{x}^i = a^i_j x^j$ in a neighborhood of the origin of the number space R^n). Now the number of eigenvalues of (a^i_j) with negative and positive real parts designates the dimensions of the stability and instability manifolds, respectively, of $\mathcal{S}$ at P. Thus no eigenvalue of $\mathcal{S}$ at P has a zero real part (that is, P is an elementary critical point of $\mathcal{S}$). Q. E. D.

The transversal germ of a periodic solution S (not a critical point) of $\mathcal{S}$ in M^n is the map of a transversal $(n - 1)$-manifold into itself which is obtained by following the solutions of $\mathcal{S}$ one circuit around a tubular neighborhood of S. After the customary equivalence identifications, the transversal germ of S in $\mathcal{S}$ is a conjugacy class in the group of germs of C^1-homeomorphisms of a neighborhood of the origin of R^{n-1} into R^{n-1}, with the origin fixed. Two periodic solutions S_1 and S_2, of differential systems $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, have tubular neighborhoods N_1 and N_2 wherein $\mathcal{S}_1$ and $\mathcal{S}_2$ are C^1-homeomorphic (with S_1 corresponding to S_2) if and only if S_1 and S_2 have the same transversal germ.

The differential of the transversal germ of S, at the origin, has eigenvalues which are the non-trivial characteristic multipliers of S. There is an oriented anchor ring tubular neighborhood of S in M^n if and only if the transversal germ is orientation preserving—otherwise there is a solid Klein bottle tubular neighborhood of S.

A periodic solution S of $\mathcal{S}$ is called isolated in case there exists a tubular neighborhood N of S in M^n such that S is the only periodic solution of $\mathcal{S}$ which lies entirely within N.

THEOREM 2. *Let S be a periodic solution of a structurally stable differential system $\mathcal{S}$ in M^n. Then S is isolated, elementary, and topologically linear.*

SKETCH OF PROOF. The last two conclusions on S mean that no (non-trivial) characteristic multiplier of S has a modulus of one; and that S has a tubular neighborhood N wherein $\mathcal{S}$ is homeomorphic with a differential system $\mathcal{S}^*$, with a corresponding periodic solution S^* in a standard anchor ring or solid Klein bottle, and S^* in $\mathcal{S}^*$ has a linear transversal germ.

Using approximation techniques, we can assume that $\mathcal{S}$ is in class C^∞ on M^n. In a tubular neighborhood N of S in M^n introduce coordinates which make N a real analytic manifold, say a solid anchor ring to simplify the exposition. Approximate $\mathcal{S}$ in N (or in the universal covering cylinder $\tilde{N}$) by a polynomialtrigonometric system

$$\mathcal{S}': \quad \dot{x}^i = \textstyle\sum_{r=0}^{s} P_r^i(x) \cos 2\pi r y + Q_r^i(x) \sin 2\pi r y$$
$$\dot{y} = 1$$

for $i = 1, 2, \cdots, n - 1$ and setting $y = x^n$. Here the real polynomials $P_r^i(x)$ and $Q_r^i(x)$ vanish at $x = 0$. Among all periodic solutions of $\mathcal{S}'$, encircling N just once, none is the limit of a sequence of isolated periodic solutions of $\mathcal{S}'$. This follows from the local arcwise connectedness of a real analytic variety [3], namely, the zeros of the square of the displacement on the $(n - 1)$-transversal through S, upon one circuit of the solutions of $\mathcal{S}'$. Thus S in $\mathcal{S}$ is either isolated or lies in a tubular subneighborhood of N which contains no isolated periodic solutions of $\mathcal{S}$ (among the periodic solutions of $\mathcal{S}$ encircling N just once).

Now select the real coefficients of $P_r^i(x)$ and $Q_r^i(x)$ in general position. Then S in $\mathcal{S}'$, and hence S in $\mathcal{S}$, is isolated among periodic solutions which encircle N just once. Moreover, S in $\mathcal{S}'$ is elementary and topologically linear [12], and the same holds for S in $\mathcal{S}$.

But then S in $\mathcal{S}$ is isolated among all periodic solutions of $\mathcal{S}$. Q. E. D.

COROLLARY. *For a given bound $T > 0$ there exist only a finite number of periodic solutions, of the structurally stable system $\mathcal{S}$ in M^n, with (minimal) periods less than T. Thus $\mathcal{S}$ has at most a countable set of periodic solutions.*

The corollary follows directly from Theorem 2 and simple continuity arguments. It is unknown whether a structurally stable system $\mathcal{S}$ on a compact manifold M^n can have an infinite number of periodic solutions. The geodesic flow in the tangent sphere bundle of a compact surface of constant negative curvature is a possible candidate for consideration.

THEOREM 3. *The positive limit set S_+ of a solution S of a structurally stable system $\mathcal{S}$ on M^n is either a critical point, a periodic solution, or each neighborhood of S_+ contains infinitely many periodic solutions of $\mathcal{S}$ with arbitrarily long (minimal) periods.*

SKETCH OF PROOF.[1] If S_+ is not a critical point or a periodic solution of $\mathcal{S}$, then S must approach a point $P \in S_+$ arbitrarily closely, and for arbitrarily large times t. Then a slight perturbation of $\mathcal{S}$ creates a differential system $\mathcal{S}'$ with a periodic solution lying in a prescribed neighborhood of S_+. By the definition of structural stability $\mathcal{S}$ must also have a periodic solution lying near S_+. Q. E. D.

COROLLARY 1. *A minimal set K of a structurally stable system $\mathcal{S}$ in M^n is either a critical point, or a periodic solution, or every neighborhood of K contains infinitely many periodic solutions of $\mathcal{S}$ with arbitrarily long (minimal) periods.*

Thus if there are only a finite number of periodic solutions of $\mathcal{S}$, the only minimal (compact, invariant) sets are critical points and periodic solutions, as conjectured by A. Andronov.

[1] A flaw has been observed in the proof of Theorem 3. Thus Theorem 3 must be considered as an additional hypothesis in Theorems 4 and 5 and the corresponding material in [7].

COROLLARY 2. *Let $\mathcal{S}$ be a structurally stable system without critical points on the compact manifold M^n. Then there exist at least two periodic solutions of $\mathcal{S}$ on M^n. If $\mathcal{S}$ has only a finite number of periodic solutions, then there must exist an orbitally stable and also an orbitally, totally unstable, periodic solution of $\mathcal{S}$.*

The existence of the orbitally stable, and unstable, periodic solutions of $\mathcal{S}$ follows from considerations of the Baire category of sets of solutions which have the periodic solutions as limit sets. It is Corollary 2 which is applicable to the Seifert problem and the Poincaré-Bendixson analogue.

For a general dynamical system [2] on a compact manifold M^n a solution is wandering in case it is embedded in a tube, which is the union of solution curves, and which never intersects itself. The complement of the wandering motions is the compact set of non-wandering motions M_1. Relative to the dynamical system restricted to M_1, define the non-wandering set M_2. Thus obtain a sequence of nested compact sets $M_1 \supset M_2 \supset \cdots$ with an intersection M_r, the central motions.

THEOREM 4. *Let $\mathcal{S}$ be a structurally stable system on a compact M^n. Then the central motions M_r are exactly the non-wandering motions M_1. Moreover, the set Σ of critical points and periodic solutions of $\mathcal{S}$ is dense in M_r, and Σ is exactly M_r in case $\mathcal{S}$ has only a finite number of periodic solutions.*

SKETCH OF PROOF. The set Σ of critical points and periodic solutions of $\mathcal{S}$ certainly lies in M_r. Now a non-wandering, non-periodic solution $S \subset M_1$ must return to a prescribed neighborhood of a point $P \in S$. Then a slight perturbation of $\mathcal{S}$ leads to the differential system $\mathcal{S}'$ with a periodic solution which passes nearly through P. Thus $\mathcal{S}$ must have a periodic solution which passes nearly through P. Therefore the closure of Σ contains M_1. Thus $M_r \supset M_1$ and $M_r = M_1$. Q. E. D.

Suppose $\mathcal{S}$ defines a volume preserving flow, say relative to a non-vanishing n-form, on the compact M^n. Then the Poincaré-Caratheodory recurrence theorem states that almost every point of M^n lies on a solution of $\mathcal{S}$ which is both $(+)$ and $(-)$ Poisson stable. Then an argument like that used in Theorem 4 yields the next result.

THEOREM 5. *Let $\mathcal{S}$ be a structurally stable system on a compact M^n and define a volume preserving flow, relative to a non-vanishing n-form. Then the union of all periodic solutions is dense in M^n.*

Thus, if $\mathcal{S}$ is structurally stable on the compact M^n and if $\mathcal{S}$ has only a finite number of periodic solutions, then $\mathcal{S}$ cannot be volume-preserving. Further, if a Hamiltonian differential system $\mathcal{H}$, say restricted to a compact energy manifold M^n, has only finitely many periodic solutions, then $\mathcal{H}$ cannot be structurally stable on M^n.

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