ON DYNAMICAL SYSTEMS*

By Stephen Smale

1. General considerations

We consider here systems (X, M) where X is a C^∞ vector field on a C^∞ closed manifold M. The vector field X generates a global 1-parameter group φ_t of differentiable homeomorphisms (or diffeomorphisms) of M. Thus for each $x \in M$, $\varphi_t(x)$ is the solution curve (or orbit) with $\varphi_0(x) = x$.

An equivalence h between (X, M) and (X', M) is a homeomorphism $h: M \to M$ which sends orbits of X into orbits of X'. Perhaps the main problem of this subject is: Given M classify all the X on M under this equivalence. It is no doubt quite unreasonable to expect an answer to this question in the foreseeable future. However, it still seems important to solve this problem for some sets C of vector fields on M, especially if C is large in some sense.

The set of all C^∞ vector fields on M with the C^1 topology (roughly X and X' are close if they are pointwise close and their first derivatives are pointwise close) form a space, say B. In view of the preceding paragraph we should look for a set $C \subset B$ where C is open and dense in B and moreover is amenable to classification in some sense. Although we are far from answering this, we would like to propose a candidate for such a C.

We say X belongs to C if it satisfies the following five conditions:

(1) There are a finite number of singular points of X, say $\beta_1, \cdots, \beta_k$, each of simple type. This means that at each β_i, the matrix of first partial derivatives of X in local coordinates has eigenvalues with real part non-zero.

(2) There are a finite number of closed orbits of X, say $\beta_{k+1}, \cdots, \beta_m$, each of simple type. This means that no characteristic exponent (see, e.g., [5]) of β_i, $i > k$, has absolute value 1.

(3) The limit points of all the orbits of X as $t \to \pm\infty$ lie on the β_i.

(4) The stable and unstable manifolds have normal intersection with each other. This can be explained as follows. Let β_i, $i \leqq k$ be one of the singular points of X, and let h be the number of eigenvalues associated to β_i with real part positive. Then [5] there is an h-dimensional C^∞ sub-manifold W_i of M passing through β_i such that if $x \in W_i$, $\lim_{t \to -\infty} \varphi_t(x) = \beta_i$. If $h = 0$ let $W_i = \beta_i$. Call W_i the unstable manifold of X at β_i.

Consider the new system X^* obtained by reversing the direction of each vector of X on M. Then β_i is a simple singularity of X^* and the above applies to yield the unstable $(n\text{-}h)$-manifold W_i^* of X^* at β_i. Call W_i^* the *stable manifold* of X at β_i.

In a similar but slightly more complicated way, one can define the stable and unstable manifolds of β_i, $i > k$, where the β_i are closed orbits. Thus for each i,

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$1 \leqq i \leqq m$, we have W_i and W_i^*, the unstable and stable manifolds of X at β_i.

For $x \in W_i$ (or W_i^*) let W_{ix} (or W_{ix}^*) be the tangent space of W_i (or W_i^*) at x. Then by the normal intersection condition we mean for each i, j, if $x \in W_i \cap W_j^*$,

$$\dim W_i + \dim W_j^* - n = \dim (W_{ix} \cap W_{jx}^*).$$

(5) If β_i is a closed orbit then there is no $y \in M$ with $\lim_{t \to -\infty} \varphi_t(y) = \beta_i$ and $\lim_{t \to \infty} \varphi_t(y) = \beta_i$.

It can be shown that these five conditions are independent. I do not think it would be difficult to show that C is open in B. However the question as to whether C is dense in B seems to be very difficult. Work of Peixoto [7] implies this is true where M is the 2 disk. The following theorem solves the corresponding approximation problem for gradient fields. (Detailed proofs of the theorems stated here will be given elsewhere.)

THEOREM 1.1. *If $X = \text{grad}\, f$, f a C^∞ function on M, then X can be C^1 approximated by a C^∞ field Y on M such that Y satisfies* (1)—(5) *with no closed orbits.*

Since there are a great variety of C^∞ functions on every closed manifold, Theorem 1.1 guarantees that for every M, C is far from being empty. The idea of the proof of 1.1 is as follows.

Given f, a theorem of Morse [6] implies there is an approximating function g on M such that g has only non-degenerate critical points. Then $Z = \text{grad}\, g$ is a C^∞ vector field on M satisfying conditions (1), (2) and (3) with no closed orbits. Furthermore any C^1 approximation of Z will have the same properties. This last fact is not true for Z in general and uses strongly the fact that Z is a gradient field. It remains to show that Z can be C^1 approximated by a field satisfying (4). To do this one changes Z using Sard's theorem [10] so that the stable and unstable manifolds fall into general position with each other.

Sections 2 and 3 contain evidence that C is amenable to classification.

2. Morse relations for X in C

For X in C let a_q be the number of β_i, $i \leqq k$ with $\dim W_i = q$, and b_q the number of β_i, $i > k$ with $\dim W_i = q$.

THEOREM 2.1. *Let $X \in C$, K be any field, R_q be the rank of $H^q(M; K)$ and $M_q = a_q + b_q + b_{q+1}$. Then M_q and R_q satisfy the Morse relations,*

$$M_0 \geqq R_0$$

$$M_1 - M_0 \geqq R_1 - R_0$$

$$\cdots$$

$$\sum_{j=0}^{n} (-1)^j M_j = (-1)^n \chi$$

where $\dim M = n$ *and χ is the Euler characteristic of M with respect to K.*

Theorem 2.1 contains theorems of Èl'sgol'c [3] and Reeb [9] and by 1.1 the classical theorem of Morse [6]. In dimension 2, Theorem 2.1 is contained in Haas [4]. We give a short sketch of the proof of 2.1 now.

Consider the sequence of closed sets L_i of M defined by $L_0 = \emptyset$, and inductively $L_i =$ union of W_j such that $\partial W_j \subset L_{i-1}$. Then strongly using conditions (3)–(5) it can be proved there is an r such that $L_r = M$. This is the hardest part of the proof, and we do not go into it here. The more obvious sequence $K_p = \bigcup W_j$, $\dim W_j \leqq p$ does not work.

Next $\Sigma_i \dim H^q(L_i, L_{i-1})$ is evaluated in Čech cohomology to be M_q. Then by a standard argument from Morse theory the theorem follows.

A problem connected with the above is the following. Let $X \in C$ with no closed orbits. In this case one can use the sequence K_p mentioned above in the proof of 2.1. Then K_p is a union of cells. Does K_p have a corresponding CW structure? If this could be shown, then probably it would lead to an intrinsic proof of the theorem that every differentiable manifold could be triangulated.

3. On structural stability

We say an equivalence is an ϵ-equivalence if it is pointwise within ϵ of the identity; let d denote a C^1 metric on B. Then $X \in B$ is *structurally stable* (according to Andronov-Pontriagen [1]) if given $\epsilon > 0$, there is a $\delta > 0$ such that if $X' \in B$, $d(X, X') < \delta$, there is an ϵ-equivalence between X and X'. Andronov and Pontriagen stated the theorem that if $M = 2$ disk, X is structurally stable if and only if X has

(1′) at most a finite number of critical points all elementary and none a center,

(2′) at most a finite number of closed paths each a limit-cycle with a nonzero characteristic number,

(3′) no separatrix joining 2 saddle points.

A proof was published by DeBaggis ([2]; see also [5]).

It is easy to see that for the 2-disk these conditions coincide with (1)–(5). Peixoto and Peixoto have extended this work to 2-manifolds and have corrected a mistake of DeBaggis.

It seems likely to us that the n-dimensional structurally stable systems are exactly the elements of C. The following problem has been considered by several people without success. Does a structurally stable system have a finite number of closed orbits? If X has only a finite number of closed orbits and is structurally stable then it must satisfy (1)–(5).

THEOREM 3.1 (L. Marcus). *If X is structurally stable and has a finite number of closed orbits then it satisfies* (1), (2), *and* (3).

THEOREM 3.2. *If X is structurally stable and has a finite number of closed orbits then it satisfies* (4) *and* (5).

The methods used to prove that X satisfies (4) do not differ very much from those used in the proof of 1.1. If X did not satisfy (4) then an arbitrarily small change of X could be made so that new intersections of stable and unstable manifolds are introduced. This is used to show that X could not have been structurally stable. To prove that X satisfies (5) one shows that if (5) is violated one can introduce new closed orbits without changing the old closed orbits by arbitrarily small changes in X. This of course is impossible if X is structurally stable.

At this time we have made a little progress on the problem as to whether conditions (1)–(5) are sufficient for structural stability.

THEOREM 3.3. *If X satisfies* (1)–(5), *there are no closed orbits, and* $\dim M \leqq 3$, *then X is structurally stable.*

The idea of the proof is to define a new structure on M depending on X. Each point of M belongs to exactly one stable manifold and one unstable manifold; hence the submanifolds $\sigma_{ij} = W_i^* \cap W_j$ as i, j range from 1 to m give a decomposition of M. Let $\Sigma^k = \{\sigma_{ij} \mid 1 \leqq i, j \leqq m, \dim \sigma_{ij} \leqq k\}$. For X' sufficiently close to X the corresponding $\Sigma^{k'}$ is related to Σ^k by an isomorphism preserving the boundary operation. Then the desired homeomorphism is defined first on Σ^0, then Σ^1, etc., by induction. The induction step poses considerable difficulties however, especially as the dimension increases.

It is not known if there exist any structurally stable systems on a given manifold. There are however examples on 2-manifolds and the n-spheres [8].

THEOREM 3.4. *There exist structurally stable systems on every closed 3-manifold.*

This follows from 1.1 and 3.3.

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