POINCARÉ'S THEOREM GENERALIZED FOR THE PERTURBATION OF INVARIANT MANIFOLDS*

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1. Introduction and summary

A famous theorem of Poincaré concerns the so-called perturbation of a periodic solution of the system,

$$\frac{dx}{dt} = f(x, t, \mu), \tag{1.1}$$

where x and f are n-vectors, t is the real independent variable, and μ is a real parameter. f is regarded as defined and sufficiently smooth in a suitable region and is periodic with some period $T > 0$ in t. Poincaré's theorem states that, if for a certain value of μ, say μ_0, the system (1.1) admits a periodic solution $x = x_0(t) = x_0(t + T)$, then it will also admit a periodic solution for all values of μ sufficiently near to μ_0, *at least if the so-called variational system,*

$$\frac{d\xi}{dt} = A(t)\xi, \quad A(t) = f_x(x_0(t), t, \mu_0), \tag{1.2}$$

has no periodic solution, other than the trivial one $\xi \equiv 0$.

We wish to obtain a similar theorem where, instead of dealing with a single solution describing the periodic motion of a single point, we shall deal with a set of many solutions describing the periodic motion of a finite manifold imbedded in our n-dimensional space. Such a manifold M_0 is regarded as moving into a transformed manifold M_t after the lapse of time t in the following manner: Consider the solution $x = x(x_0, t, \mu)$ of (1.1) such that $x(x_0, 0, \mu) = x_0$. Then, fixing t and μ, and allowing x_0 to vary over M_0, the locus of the point $x = x(x_0, t, \mu)$ is, by definition, the manifold M_t referred to above.

Now, if it happens that M_T is the same as the initial manifold M_0, we shall say that M_0 or M_t possesses a periodic motion, or, more briefly, that M_t is periodic. This can happen even though few or none of the individual solution curves along which the points of M_t move are periodic. For we, of course, do not require that $x(x_0, T, \mu) = x_0 \in M_0$, but merely that $(x_0 \in M_0) \Rightarrow (x(x_0, T, \mu) \in M_0)$.

The initial manifold M_0 can be described, at least locally, by means of parametric equations $x = x(s)$, where $s = (s_1, \cdots, s_k)$, k being the dimensionality of the manifold. It is hereby assumed that the $x(s)$'s are at least differentiable (when $\mu = \mu_0$) and that the Jacobian matrix $\partial x/\partial s$ is of rank k. In this connec-

* This research was partially supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract Number AF 49(638)-382. Reproduction in whole or in part is permitted for any purpose of the United States Government.

tion, it should be noted that it is not necessary to have a uniform global representation for either the manifolds which we discuss or for the differential equations, even though this is certainly possible in the simpler special cases. We can always use local representations valid in a finite number of overlapping neighborhoods and the appropriate transformations in passing from one neighborhood to an overlapping neighborhood.

In the sequel it will be convenient to choose the coordinates in each neighborhood representation of the system (1.1) so that $x_{n-k+1} = s_1$, $x_{n-k+2} = s_2, \cdots,$ $x_n = s_k$. Thus the system (1.1), in a somewhat different notation, appears in the form,

$$\frac{dx}{dt} = f(x, s, t, \mu) \tag{1.3}$$

$$\frac{ds}{dt} = g(x, s, t, \mu), \tag{1.4}$$

where now x and f are m-vectors, $m = n - k$, while s and g are k-vectors. Both f and g are periodic in t with period T.

Perhaps a better way of interpreting equations (1.3) and (1.4) is to regard s abstractly as representing a point on M_0, or any fixed k-manifold $\mathfrak{M}$ homeomorphic to M_0, while x is to be regarded abstractly as a point in some m dimensional linear space $\mathfrak{R}$. The equations (1.3) and (1.4) then determine the motion of the point (s, x) in the Cartesian product space $\mathfrak{M} \times \mathfrak{R}$. It is assumed that in a neighborhood of any point in $\mathfrak{M} \times \mathfrak{R}$, the equations (1.3) and (1.4) can actually be written down in the usual way in terms of coordinates representing both s and x, and that only a finite number of such neighborhoods suffice for the study of all motions of interest.

If the equation of the manifold M_t is written in the form,

$$x = u(s, t, \mu), \tag{1.5}$$

where u is an m-vector, it means that, $(s(t), x(t))$ being any solution of (1.3) and (1.4) which starts from an initial point (s_0, x_0) on M_0, we must have $x(t) \equiv u(s(t), t, \mu)$. Differentiating this identity totally with respect to t and using (1.3) and (1.4) to eliminate the derivatives of x and s, we find that u must satisfy the partial differential equation

$$\frac{\partial u}{\partial t} = f(u, s, t, \mu) - \frac{\partial u}{\partial s} g(u, s, t, \mu). \tag{1.6}$$

Conversely the solution $u(s, t, \mu)$ of (1.6), such that $x = u(s, 0, \mu)$ is the equation of M_0, is seen to furnish the manifold M_t via (1.5). The problem of finding a periodic manifold M_0 of (1.3) and (1.4) is thus seen to be equivalent to the problem of finding a periodic solution $u(s, t, \mu)$ of (1.6), with period T in t, at least under the assumption that u is of class C'.

Suppose such a periodic solution $u_0(s, t)$ is known for a certain value μ_0 of μ.

Relative to this known periodic solution, we write down a system of linear partial differential equations which bear the same relation to (1.6) that the variational equations (1.2) bear to (1.1). We therefore call them the variational system for our manifold problem. They are

$$\frac{\partial w}{\partial t} = A(s, t)w - \frac{\partial w}{\partial s} G(s, t) \tag{1.7}$$

where

$$A(s, t) = f_x(u_0(s, t), s, t, \mu_0) - \frac{\partial u_0(s, t)}{\partial s} g_x(u_0(s, t), s, t, \mu_0). \tag{1.8}$$

and

$$G(s, t) = g(v(s, t), s, t, \mu). \tag{1.9}$$

Here, of course, f_x is an $m \times m$ matrix, $\partial u_0/\partial s$ an $m \times k$ matrix, and g_x a $k \times m$ matrix. $v(s, t)$ is an arbitrary m-vector function of sufficient regularity close to $u_0(s, t)$, and $\mu - \mu_0$ is also to be regarded as sufficiently small.

It is to be noticed that the problem treated by Poincaré may be regarded as a special case of the present problem with $k = 0$. Hence one might, by extrapolating from Poincaré's result, hazard the conjecture that, if (1.6) possesses a periodic solution for a particular value μ_0 of μ, it will continue to have a periodic solution for $\mu \neq \mu_0$ but sufficiently close to μ_0, *at least provided that the system* (1.7) *has no periodic solution other than the trivial one* $w \equiv 0$. We would in this way obtain a sufficient condition for the existence of periodic manifolds which is a direct generalization of Poincaré's classical theorem on the perturbation of periodic solutions. The purpose of this paper was to investigate the possibility of obtaining such a theorem. Our main result is to the effect that such a theorem is indeed true if we modify the italicized proviso to read as follows: *at least provided that the system* (1.7) *does not "come close" to having a non-trivial periodic solution and provided that a certain linear transformation* T_v *defined in certain Banach spaces should have a sufficiently extensive range*. The meaning of the phrase "come close" will be explained in detail later on. It is introduced to insure that T_v should have a bounded inverse.

In some recently published work, Walter T. Kyner also has pointed out the importance of assuming the boundedness of the inverse of a somewhat similar transformation. This work of Kyner was invaluable in giving insight as to what should reasonably be expected in the present more general situation.

Suppose we begin by noting (with the proof deferred to Section 2) that the equation (1.6) can be written in the form

$$\frac{\partial w}{\partial t} = A(s, t)w - \frac{\partial w}{\partial s} g(u_0 + w, s, t, \mu) + \Phi(u_0 + w, s, t, \mu), \tag{1.10}$$

where $u = u_0 + w$, and where Φ vanishes to the second order in w and $(\mu - \mu_0)^{1/2}$, and where $u_0 = u_0(s, t)$ is, of course, the known solution of (1.6), when $\mu = \mu_0$, as previously mentioned. This suggests that, for purposes of successive approximations, we study the non-homogeneous linear system

$$\frac{\partial w}{\partial t} = A(s, t)w - \frac{\partial w}{\partial s} g(v(s, t), s, t, \mu) + \varphi(s, t), \tag{1.11}$$

where $v(s, t)$ and $\varphi(s, t)$ are known m-vector functions, periodic in t with period T and of class C^{r}. If $\varphi(x, t) \equiv 0$, this is the variational equation (1.7) previously introduced.

Suppose that $s = S_v(s_0, t_0, t)$ be the solution of the system,

$$\frac{ds}{dt} = g(v(s, t), s, t, \mu), \tag{1.12}$$

of order k, which satisfies the initial conditions,

$$S_v(s_0, t_0, t_0) = s_0 . \tag{1.13}$$

In case $v(s, t) = u_0(s, t)$, which gives the known periodic manifold when $\mu = \mu_0$, we know that for $\mu = \mu_0$, the solution $S_v(s_0, t_0, t)$ is defined for all t. Hence if $|\mu - \mu_0|$ and $\max |v(s, t) - u_0(s, t)|$ are sufficiently small, $S_v(s_0, t_0, t)$ will be defined for $|t - t_0| < 2T$.

From the theory of linear homogeneous ordinary differential equations, we know that there exists an $m \times m$ matrix $\Omega_v(s_0, t_0, t)$ such that

$$\frac{\partial \Omega_v}{\partial t} = A[S_v(s_0, t_0, t), t]\Omega_v \tag{1.14}$$

and

$$\Omega_v(s_0, t_0, t_0) = I, \tag{1.15}$$

the identity matrix. We are now in a position to state the

THEOREM. *The* (unique) *solution of* (1.11) *which satisfies the condition*

$$w(s, 0) = \alpha(s) \tag{1.16}$$

may be written in the form

$$\begin{aligned} w(s, t) &= \Omega_v[S_v(s, t, 0), 0, t]\alpha[S_v(s, t, 0)] \\ &\quad + \int_0^t \Omega_v[S_v(s, t, \tau), \tau, t]\varphi[S_v(s, t, \tau), \tau]\, d\tau. \end{aligned} \tag{1.17}$$

The proof of this theorem is straightforward and will be given in Section 2. However, instead of the initial problem posed by (1.16), we wish to get a *periodic* solution. This means that $w(s, T) = \alpha(s)$. Hence we find that $\alpha(s)$ is to satisfy the condition

$$\alpha(s) - \Omega_v[S_v(s, T, 0), 0, T]\alpha[S_v(s, T, 0)] = \int_0^T \Omega_v(S_v(s, T, \tau), \tau, T)\varphi(*)\, d\tau.$$

Thus to solve for α we need to invert the linear transformation T_v defined as follows:

$$T_v\alpha(s) = \alpha(s) - \Omega_v[S_v(s, T, 0), 0, T]\alpha[S_v(s, T, 0)]. \tag{1.18}$$

The mere existence of T_v^{-1} is insured if we assume that (1.11) has no nontrivial periodic solution in the homogeneous case $\varphi \equiv 0$. But this would be ineffective if the domain of T_v^{-1}, that is, the range of T_v, did not contain the right hand member β of the equation $T_v\alpha = \beta$ to be solved for α. Moreover, in order to employ our projected system of successive approximations or, what in this case amounts to almost the same thing, the Schauder fixed point theorem, we have found it necessary (under the general situation here considered) to assume that T_v^{-1} is uniformly bounded for $| v - u_0 |$ sufficiently small. That is, there exists a constant C such that

$$\| T_v^{-1}\alpha \| \leqq C \| \alpha \|. \tag{1.19}$$

The question now arises as to the definition of $\| \alpha \|$ and as to the particular Banach space to be considered. Under the general conditions here considered, it seems necessary to consider *two* Banach spaces $\bar{B}_0$ and $\bar{B}_1$, in both of which T_v is defined, namely: $\bar{B}_0$ = space of m-vector functions $\alpha(s)$ continuous over $\mathfrak{M}$ and with norm $\| \alpha \|_{\bar{B}_0} = \max_{s,i} | \alpha_i(s)|$. $\bar{B}_1$ = space of m-vector functions $\alpha(s)$ of class C' over $\mathfrak{M}$ and with norm $\| \alpha \|_{\bar{B}_1} = \max_{s,i,j} \{| \alpha_i(s)|, | \alpha'_{ij}(s)|\} \cdot \alpha'_{ij}(s) = (d\alpha_i(s)/ds_j)$. (1.19) is then to be interpreted as meaning that both $\| T_v^{-1}\alpha \|_{\bar{B}_0} \leqq C \| \alpha \|_{\bar{B}_0}$ if $\alpha \in \bar{B}_0$ and also $\| T_v^{-1}\alpha \|_{\bar{B}_1} \leqq C \| \alpha \|_{\bar{B}_1}$ if α is also $\in \bar{B}_1$. This means that the linear "variational" equation

$$\frac{\partial w}{\partial t} = A(s, t)w - \frac{\partial w}{\partial s} g(v(s, t), s, t, \mu) \tag{1.20}$$

does not "come close" to having a periodic solution in the following sense: There exists a number $\epsilon > 0$, such that there is no solution of (1.20), normalized by the condition,

$$\| w(s, 0)\| = 1, \quad \text{such that} \quad \| w(s, 0) - w(s, T)\| < \epsilon.$$

Here the norms are to be taken in both senses, and we may take $\epsilon = C^{-1}$.

Finally it should be mentioned that the conditions here imposed seem to be satisfied in all the specialized cases previously treated.

2. Proofs of some of the more elementary statements made in section 1

In establishing that (1.6) can be written in the form (1.10), we recall the supposition that equations (1.3) and (1.4) have a known periodic manifold, $x = u_0(s, t)$. In other words, u_0 is a periodic solution of (1.6) when $\mu = \mu_0$; so that we have

$$\frac{\partial u_0}{\partial t} = f(u_0, s, t, \mu_0) - \frac{\partial u_0}{\partial s} g(u_0, s, t, \mu_0).$$

Subtracting this identity from (1.6) and writing $w = u - u_0$, we find that

$$\frac{\partial w}{\partial t} = [f(u, s, t, \mu) - f(u_0, s, t, \mu_0)] - \frac{\partial u_0}{\partial s}[g(u, s, t, \mu) - g(u_0, s, t, \mu_0)] - \frac{\partial w}{\partial s} g(u, s, t, \mu). \tag{2.1}$$

Define $\varphi(u, s, t, \mu)$ and $\psi(u, s, t, \mu)$ as follows:

$$\varphi(u, s, t, \mu) = f(u, s, t, \mu) - f(u_0, s, t, \mu_0) - f_u(u_0, s, t, \mu_0)(u - u_0)$$

$$\psi(u, s, t, \mu) = g(u, s, t, \mu) - g(u_0, s, t, \mu_0) - g_u(u_0, s, t, \mu_0)(u - u_0),$$

so that φ and ψ together with their derivatives with respect to u vanish when $u = u_0$ and $\mu = \mu_0$. Making the appropriate substitution in (2.1), we easily find that the latter appears in the form

$$\frac{\partial w}{\partial t} = \varphi(u, s, t, \mu) + f_u(u_0, s, t, \mu_0)w - \frac{\partial u_0}{\partial s}[\psi(u, s, t, \mu) + g_u(u_0, s, t, \mu_0)w] - \frac{\partial w}{\partial s} g(u, s, t, \mu). \tag{2.2}$$

Finally, introducing the matrix $A(s, t)$ defined by (1.8) and also the vector $\Phi(u, s, t, \mu) = \varphi(u, s, t, \mu) - (\partial u_0/\partial s)\psi(u, s, t, \mu)$, we find at once that the equation (2.2) may be written in the form (1.10), as we wished to prove. Φ, of course, has the property that it vanishes together with its derivatives with respect to u when $u = u_0$ and $\mu = \mu_0$. In verifying these statements the reader should note that the f_x and g_x of (1.8) mean the same thing as the f_u and g_u of (2.2).

The problem of solving (1.11) under the initial conditions $w(s, 0) = \alpha(s)$ is the same as that of determining the manifold of trajectories of the system

$$\dot{x} = A(s, t)x + \varphi(s, t) \tag{2.3}$$

$$\dot{s} = g(v(s, t), s, t, \mu) = g[s, t] \tag{2.4}$$

issuing from the manifold $x = \alpha(s)$ when $t = 0$. In fact, the relationship between (2.3), (2.4), and (1.11) is precisely the same as the relationship, in the more general case, between (1.3), (1.4), and (1.6), except that we now write w instead of u. In order to prove our theorem we begin by considering the general system $\dot{x} = f(x, s, t)$, $\dot{s} = g(x, s, t)$ and its general solution,

$$x = X(x_0, s_0, t_0, t), \qquad s = S(x_0, s_0, t_0, t), \tag{2.5}$$

taking on the initial values x_0, s_0, when $t = t_0$. Evidently, since (x, s) and (x_0, s_0) in (2.5) represent a pair of points on the same trajectory of the system at "times" t and t_0 respectively, there is complete symmetry between (x, s, t) and (x_0, s_0, t_0). Hence (2.5) is easily solved for x_0 and s_0 in terms of x, s, t, and t_0 in the form,

$$x_0 = X(x, s, t, t_0) \tag{2.6}$$

$$s_0 = S(x, s, t, t_0). \tag{2.7}$$

Suppose now that $x_0 = \alpha(s_0)$. Then from (2.5) and (2.7) we obtain

$$x = X(x_0, s_0, t_0, t) = X(\alpha(s_0), s_0, t_0, t)$$

$$x = X(\alpha(S(x, s, t, t_0)), S(x, s, t, t_0), t_0, t).$$

With $t_0 = 0$, $x = w(s, t)$, this becomes

$$w(s, t) = X(\alpha(S(x, s, t, 0)), S(x, s, t, 0), 0, t), \tag{2.8}$$

and, of course when $t = 0$ also, we get

$$\begin{aligned} w(s, 0) &= X(\alpha(S(x, s, 0, 0)), S(x, s, 0, 0), 0, 0) \\ &= \alpha(S(x, s, 0, 0)) = \alpha(s), \end{aligned} \tag{2.9}$$

as we should. In obtaining these last reductions we use the identities $X(x, s, t, t) = x$ and $S(x, s, t, t) = s$ which are implicit in the definition of X and S.

We next consider the special case with which we are primarily concerned. That is, we now assume that $f(x, s, t) = A(s, t)x + \varphi(s, t)$ while $g(x, s, t) = g[s, t]$. Thus, in accordance with (1.12) we may write $S(x_0, s_0, t_0, t) = S_v(s_0, t_0, t)$, obtaining a simplification from the fact that S in this special case is independent of x_0. We also get an explicit expression for $X(x_0, s_0, t_0, t)$ in terms of the matrix $\Omega_v(s_0, t_0, t)$ and the Lagrange "variation-of-parameters" formula. Namely,

$$\begin{aligned} X(x_0, s_0, t_0, t) &= \Omega_v(s_0, t_0, t)x_0 \\ &+ \int_{t_0}^{t} \Omega_v(s_0, t_0, t)\Omega_v(s_0, t_0, \tau)^{-1} \varphi[S_v(s_0, t_0, \tau), \tau] \, d\tau. \end{aligned} \tag{2.10}$$

In order to simplify this formula, we shall first prove some properties of S_v and Ω_v, namely

$$S_v(s_0, t_0, t) \equiv S_v(S_v(s_0, t_0, \tau), \tau, t) \tag{2.11}$$

and

$$\Omega_v(s_0, t_0, t)\Omega_v(s_0, t_0, \tau)^{-1} \equiv \Omega_v(S_v(s_0, t_0, \tau), \tau, t). \tag{2.12}$$

We obtain (2.11) by considering three points s_0, σ, and s on the same trajectory of the system $\dot{s} = g[s, t]$ at times t_0, τ, and t respectively. Evidently, we must have $s = S_v(s_0, t_0, t) = S_v(\sigma, \tau, t)$, while $\sigma = S_v(s_0, t_0, \tau)$. Inserting this value of σ in the previous equation, we get (2.11) immediately.

We next notice from (2.11) that the equation

$$\frac{d\Omega}{dt} = A[S_v(s_0, t_0, t), t] \, \Omega \tag{2.13}$$

can also be written in the form,

$$\frac{d\Omega}{dt} = A[S_v(\sigma, \tau, t), t]\,\Omega, \tag{2.14}$$

where $\sigma = S_v(s_0, t_0, \tau)$ as before. Now, if we let $\Omega = \Omega_v(s_0, t_0, t)\Omega_v(s_0, t_0, \tau)^{-1}$, we see from (1.14) that (2.13) is satisfied, and hence (2.14) is also satisfied. Furthermore, when $t = \tau$, the second factor in the expression for Ω is the inverse of the first factor, and so Ω reduces to I. From the uniqueness property of the system (2.14) together with the definition of $\Omega_v(\sigma, \tau, t)$ (cf. (1.14) and (1.15) with s_0, t_0 replaced by σ, τ), we see therefore that $\Omega = \Omega_v(\sigma, \tau, t)$, which establishes (2.12).

Evidently with the help of (2.12), we may write (2.10) in the form

$$\begin{aligned} X(x_0, s_0, t_0, t) &= \Omega_v(s_0, t_0, t)x_0 \\ &\quad + \int_{t_0}^{t} \Omega_v(S_v(s_0, t_0, \tau), \tau, t)\Phi[S_v(s_0, t_0, \tau), \tau]\, d\tau. \end{aligned} \tag{2.15}$$

This is the particular expression for X, which we wish to use with (2.8), wherein we must replace x_0 by $\alpha(S_v(s, t, 0))$, s_0 by $S_v(s, t, 0)$, and t_0 by 0. We thus obtain

$$\begin{aligned} x &= \Omega_v(S_v(s, t, 0), 0, t)\alpha(S_v(s, t, 0)) \\ &\quad + \int_0^t \Omega_v(S_v(S_v(s, t, 0), 0, \tau), \tau, t)\varphi[S_v(S_v(s, t, 0), 0, \tau), \tau]\, d\tau. \end{aligned} \tag{2.16}$$

From (2.11), it is seen that the right hand side of (2.16) reduces to the right hand side of (1.17).

In this proof of the theorem of Section 1, the reader should notice that no use was made of the partial differential equation (1.11) except for the statement concerning the general equivalence between the solution of the partial differential equation (1.6) and the determination of the k-manifolds of solutions of (1.3) and (1.4). This equivalence is, in fact, not entirely valid unless the k-manifolds in question are assumed to be differentiable. Moreover, the function $w(s, t)$ given by (1.17) is *not*, in general, a solution of (1.11) in the ordinary sense unless $\alpha(s)$ is differentiable. If $\alpha(s)$ is merely required to be continuous, $w(s, t)$ can, however, still be regarded as a solution of (1.11) in various familiar generalized senses, some detailed references to which will be found in the bibliography. Furthermore, the partial differential equations are actually bypassed (as in the above proof) in all of our really essential considerations. They are introduced for heuristic and interpretive reasons. Hence we shall sometimes and without further comment not require $\alpha(s)$ and $w(s, t)$ to be differentiable.

3. The existence proof

We have already defined the linear transformation T_v by (1.18). The domain $\bar{B}_0$ of this transformation T_v is the set of continuous functions defined on the

k-manifold $\mathfrak{M}$ with values in the m-dimensional space $\mathfrak{R}$, and its range is also in $\bar{B}_0$. We must also introduce four other linear transformations L_v, L_v^{-1}, K_v, and H_v. The domains of these four transformations are all the same, namely the set B_0 of continuous functions defined on the product space $\mathfrak{M} \times [0, T]$ with values in $\mathfrak{R}$. Here $[0, T]$ is the closed interval from 0 to T. The ranges of L_v, L_v^{-1}, K_v are also in B_0, but the range of H_v is to be $\subset \bar{B}_0$. With these understandings, our four transformations are defined by the following formulas:

$$[L_v w](s, t) = \Omega_v[S_v(s, t, 0), 0, t]w[S_v(s, t, 0), t] \tag{3.1}$$

$$[L_v^{-1} w](s, t) = \Omega_v[S_v(s, 0, t), t, 0]w[S_v(s, 0, t), t] \tag{3.2}$$

$$[K_v\varphi](s, t) = \int_0^t \Omega_v[S_v(,s\, t, \tau), \tau, t]\varphi[S_v(s, t, \tau), \tau]\, d\tau \tag{3.3}$$

$$[H_v\varphi](s) = [K_v\varphi](s, T). \tag{3.4}$$

The fact that L_v and L_v^{-1} are inverses of each other is easily established from (2.11) and (2.12).

$\bar{B}_0$ may be thought of as a subset of B_0. Hence any of the above four transformations make sense when applied to an element w in $\bar{B}_0$, that is, when w is independent of t. When we wish to emphasize or specify this restricted use of a transformation, we will use a superscript bar. Thus $\bar{L}_v$, for example, has domain $\bar{B}_0$ but its range is a subset of B_0 which, in general, is quite distinct from $\bar{B}_0$.

It is clear from (1.18) and (3.1) that T_v and $\bar{L}_v$ are somewhat related to each other. Namely,

$$[T_v\alpha](s) = \alpha(s) - (\bar{L}_v\alpha)(s, T). \tag{3.5}$$

We may now write (1.17) in the condensed form

$$w = \bar{L}_v\alpha + K_v\varphi \tag{3.6}$$

Since $w(s, 0) = \alpha(s)$ by the theorem of Section 1, the condition for periodicity is $\alpha(s) = w(s, T)$; that is, with the help of (3.5), (3.6) and (3.4), $T_v\alpha = H_v\varphi$. Assuming that T_v^{-1} exists, we may solve this equation for α and insert its value in (3.6), thus obtaining

$$w = (\bar{L}_v T_v^{-1} H_v + K_v)\varphi, \tag{3.7}$$

as the formula for the (unique) periodic solution of (1.11), which must, of course, exist if T_v^{-1} exists.

Comparing (1.10) with (1.11) and remembering that the periodic solution of the latter is given by (3.7), it is clear that for $w(s, t)$ to be a periodic solution of (1.10) it is necessary and sufficient for

$$w(s, t) = (\bar{L}_u T_u^{-1} H_u + K_u)\Phi(u_0(s, t) + w(s, t), s, t, \mu), \tag{3.8}$$

where, of course, the right hand member depends on $w(s, t)$ not only because w appears as an argument of Φ but also because the subscript u is an abbreviation for $u_0(s, t) + w(s, t)$. It should also be remembered that L, T, H, and K all depend upon μ even though, for simplicity in notation, this is not indicated in (3.8).

The right hand member of (3.8) defines a non-linear operator G_μ, depending upon μ, so that the solution of (3.8) amounts to finding a fixed point of the transformation,

$$\bar{w} = G_\mu w = (\bar{L}_u T_u^{-1} H_u + K_u)\Phi(w) \tag{3.8'}$$

where w and $\bar{w}$ are suitable elements of B_0 and $\Phi(w)$ is written as an abbreviation for $\Phi(u_0 + w, s, t, \mu)$. We propose to show that a fixed point always exists for sufficiently small $|\mu - \mu_0|$ by appealing to the Schauder fixed point theorem.

The Banach spaces $\bar{B}_0$ and $\bar{B}_1$ have already been introduced together with the appropriate norms $\|\alpha\|_{\bar{B}_0}$ and $\|\alpha\|_{\bar{B}_1}$. We have also already introduced B_0 as the set of continuous m-vector functions defined on $\mathfrak{M} \times [0, T]$. We now introduce B_1 as the subset of B_0 possessing continuous derivatives, and we then introduce suitable norms for both B_0 and B_1, thus allowing us to consider them as Banach spaces. These norms are analogous to those introduced in Section 1 for $\bar{B}_0$ and $\bar{B}_1$ and are defined as follows:

$$\|w\|_{B_0} = \max_{s,i,t} |w_i(s,t)|$$

$$\|w\|_{B_1} = \max_{s,i,j,t} \left\{ |w_i(s,t)|, \left|\frac{\partial w_i}{\partial s_j}\right| \right\}$$

The linear transformations L_v, L_v^{-1}, K_v, $\bar{L}_v$, H_v, T_v are of sufficiently simple form so that the reader may readily verify that they are all bounded with respect to either appropriate norm, at least if $|\mu - \mu_0|$ and $\|v - u_0\|$ are sufficiently small. More precisely, we assert that there exist positive numbers δ and C such that $|\mu - \mu_0| < \delta$, $\|v - u_0\|_{Bi} < \delta$ (whether $i = 0$ or 1) imply that

$$\begin{aligned} &\|L_v w\|_{B_i} \leqq C\|w\|_{B_i} && \text{for every } w \in B_i \\ &\|L_v^{-1} w\|_{B_i} \leqq C\|w\|_{B_i} && \text{for every } w \in B_i \\ &\|K_v w\|_{B_i} \leqq C\|w\|_{B_i} && \text{for every } w \in B_i \\ &\|\bar{L}_v \alpha\|_{B_i} \leqq C\|\alpha\|_{\bar{B}_i} && \text{for every } \alpha \in \bar{B}_i \\ &\|H_v w\|_{\bar{B}_i} \leqq C\|w\|_{B_i} && \text{for every } w \in B_i \\ &\|T_v \alpha\|_{\bar{B}_i} \leqq C\|\alpha\|_{\bar{B}_i} && \text{for every } \alpha \in \bar{B}_i \end{aligned} \tag{3.9}$$

We need to add to this list the additional inequality

$$\|T_v^{-1}\alpha\|_{\bar{B}_i} \leqq C\|\alpha\|_{\bar{B}_i} \qquad \text{for every } \alpha \in \bar{B}_i. \tag{3.10}$$

Unlike (3.9), it is not possible to prove (3.10). As already indicated in Section 1 (cf. (1.19) *et seq.*), we introduce (3.10) as our major hypothesis.

Another property of our transformations to be needed in the sequel is formulated in the following

LEMMA. *Let w^* be any fixed element of B_0 and let α^* be any fixed element of $\bar{B}_0$. Let $\| w_h \|_{B_0} < \delta$ for $h = 1, 2, \cdots$ and let $w_h \to w$ uniformly (in other words, $\| w_h - w \|_{B_0} \to 0$) and let us write $u_h = u_0 + w_h$, $u = u_0 + w$. Then, in the sense of uniform convergence, the following limiting relationships are valid:*

$$L_{u_h}^{-1} w^* \to L_u^{-1} w^*, \qquad L_{u_h}^{-1} K_{u_h} w^* \to L_u^{-1} K_u w^*,$$
$$H_{u_h} w^* \to H_u w^*, \qquad T_{u_h} \alpha^* \to T_u \alpha^*.$$

The proof of this lemma is left to the reader, but we remark that the proof involves the uniform continuity of w^* on $\mathfrak{M} \times [0, T]$ and of α^* on $\mathfrak{M}$. This is justified, since it has been assumed from the outset that $\mathfrak{M}$ (or M_0) is a finite manifold and is therefore compact.

We next choose a positive number $\theta \leqq \delta$ so small that

$$\| \Phi(w) \|_{B_i} < \theta(C^3 + C)^{-1}, \qquad i = 0, 1, \tag{3.11}$$

whenever $w \in B_i$ and $\| w \|_{B_i} \leqq \theta$ and $| \mu - \mu_0 |^{1/2} < \theta$. We can do this because Φ vanishes to the second order in w and $(\mu - \mu_0)^{1/2}$. Hence from (3.8′) and the triangle inequality we find that

$$\| \bar{w} \|_{B_i} \leqq \| L_u T_u^{-1} H_u \Phi(w) \|_{Bi} + \| K_u \Phi(w) \|_{B_i}$$

and thus from (3.9), (3.10) and (3.11) we find further that

$$\| w \|_{B_i} \leqq C^3 \theta (C^3 + C)^{-1} + C\theta(C^3 + C)^{-1} = \theta.$$

Hence our transformation G_μ, for any fixed value of μ differing from μ_0 by less than θ, sends the convex set $\mathcal{S} = (\| w \|_{B_0} \leqq \theta) \cap (\| w \|_{B_1} \leqq \theta)$ into itself. This set is not compact or even conditionally compact in the norm for B_1. It is, however, conditionally compact in the norm for B_0, a fact which is an immediate consequence of Ascoli's theorem and the fact that the derivatives of w exist and are bounded throughout the set $\| w \|_{B_1} \leqq \theta$. All functions belonging to the closure $\bar{\mathcal{S}}$ of $\mathcal{S}$ relative to B_0 thus satisfy a Lipschitz condition with Lipschitz constant equal to θ.

We next prove that G_μ sends any element of $\bar{\mathcal{S}}$ into $\bar{\mathcal{S}}$. In the first place it is obvious that $G_\mu w$ is well defined for $w \in \bar{\mathcal{S}}$, since $\bar{\mathcal{S}} \subset B_0$. Assuming then that $w \in \bar{\mathcal{S}}$, there exists a sequence $w_h \in \mathcal{S} (h = 1, 2, \cdots)$ such that $\| w_h - w \|_{B_0} \to 0$. If then we write $\bar{w}_h = G w_h$, we know from the compactness of $\bar{\mathcal{S}}$ that there exists an element $\bar{w} \in \bar{\mathcal{S}} \ni \| \bar{w}_h - \bar{w} \|_{B0} \to 0$, at least, if we confine attention to a suitable subsequence. We also note that $\| \Phi(w_h) - \Phi(w) \|_{B_0} \to 0$. We wish to prove that $\bar{w} = G_\mu(w)$.

By definition of G_μ we evidently have

$$\bar{w}_h = L_{u_h} T_{u_h}^{-1} H_{u_h} \Phi(w_h) + K_{u_h} \Phi(w_h), \qquad \text{where } u_h = u_0 + w_h .$$

Hence

$$T_{u_h}^{-1} H_{u_h} \Phi(w_h) = L_{u_h}^{-1} \bar{w} - L_{u_h}^{-1} K_{u_h} \Phi(w_h). \tag{3.12}$$

Also, from the triangle inequality, we have

$$\| L_{u_h}^{-1}\bar{w}_h - L_u^{-1}\bar{w} \|_{B_0} \leqq \| L_{u_h}^{-1}\bar{w}_h - L_{u_h}^{-1}\bar{w} \|_{B_0} + \| L_{u_h}^{-1}\bar{w} - L_u^{-1}\bar{w} \|_{B_0} .$$

The second term on the right tends to 0 as $h \to \infty$ in accordance with our Lemma (taking $\bar{w} = w^*$). The first term on the right, in accordance with (3.9), does not exceed $C \| \bar{w}_h - \bar{w} \|_{B0}$ which also tends to 0. Hence $L_{u_h}^{-1} \bar{w}_h \to L_u^{-1}\bar{w}$. In a similar way, we use our Lemma to prove that $L_{u_h}^{-1}K_{u_h}\Phi(w_h) \to L_u^{-1}K_u\Phi(w)$. Observing that the left member of (3.12) $\in \bar{B}_0$ (i.e., it is independent of t), the same is true of the right member and of the limit of the right member, namely $L_u^{-1}\bar{w} - L_u^{-1}K_u\Phi(w)$ even though the individual terms are not in general elements of $\bar{B}_0$ but only of B_0. It follows that there exists $\zeta_h \in \bar{B}_0 \ni \zeta_h \to 0$ and $T_{u_h}^{-1}H_{u_h}\Phi(w_h) = L_u^{-1}\bar{w} - L_u^{-1}K_u\Phi(w) + \zeta_h$. From this we obtain $H_{u_h}\Phi(w_h) = T_{u_h}[L_u^{-1}\bar{w} - L_u^{-1}K_u\Phi(w)] + T_{u_h}\zeta_g$. Again with the help of our Lemma we prove that $H_{u_h}\Phi(w_h) \to H_u\Phi(w)$ and that

$$T_{u_h}[L_u^{-1}\bar{w} - L_u^{-1}K_u\Phi(w)] \to T_u[L_u^{-1}\bar{w} - L_u^{-1}K_u\Phi(w)].$$

Moreover by (3.9) $\| T_{u_h}\zeta_h \|_{\bar{B}_0} \leqq C \| \zeta_h \|_{\bar{B}_0} \to 0$. It follows that $H_u\Phi(w) = T_u[L_u^{-1}\bar{w} - L_u^{-1}K_u\Phi(w)]$. From this we easily prove the desired result, namely $\bar{w} = L_uT_u^{-1}H_u\Phi(w) + K_u\Phi(w) = G_\mu w$. Since $G_\mu w$ is single valued, it is clear that $\bar{w}$ is the *only* limit point of the original sequence $\bar{w}_h$ introduced above; so that, a posteriori, we find that it would have been unnecessary to confine attention to a subsequence. By taking $w_h \in \bar{\mathfrak{S}}$ instead of in $\mathfrak{S}$, the same argument shows that G_μ is continuous in $\bar{\mathfrak{S}}$.

Thus we have proved that for $| \mu - \mu_0 | < \theta$ the transformation $\bar{w} = G_\mu(w)$ takes the convex compact set $\bar{\mathfrak{S}} \subset B_0$ into itself and moreover is continuous. Hence, by the Schauder fixed point theorem, for each such fixed value of μ, there exists $w = w(s, t; \mu) \in \mathfrak{S} \subset B_0$, such that $w = G_\mu(w)$. This $w(s, t; \mu)$ must then furnish a solution to the problem proposed in Section 1 of finding an invariant or periodic manifold for the system (1.3), (1.4). We thus have established the following

THEOREM. *Let $f(x, s, t; \mu)$ and $g(x, s, t; \mu)$ be of class C' and suppose that the system* (1.3), (1.4), *admits a periodic manifold $x = u_0(s, t)$ of class C' when $\mu = \mu_0$. Suppose furthermore that the variational equation* (1.20) *does not "come close" (uniformly in v) to having a periodic solution, in the sense that $\| T_v^{-1}\alpha \|_{\bar{B}_1} \leqq C \| \alpha \|_{\bar{B}_i}$ for every $\alpha \in \bar{B}_i$, for some constant C and any $v \in B_i$ such that $\| v - u_0 \|_{B_i} < \delta$, for $i = 0$ and for $i = 1$. (This implies the assumption that the range of T_v is $\bar{B}_i$.) Then there exists a positive number $\theta \leqq \delta$, such that the system* (1.3), (1.4), *possesses a periodic manifold for each value of μ satisfying the inequality $| \mu - \mu_0 | < \theta$. Moreover this manifold is given by an equation of the form $x = u_0 + w(s, t; \mu)$ where w is continuous in s and t and $\| w(s, t; \mu) \|_{B_0} \leqq \theta$. Furthermore w satisfies Lipschitz conditions with respect to s and t with Lipschitz constant $= \theta$.*

4. Uniqueness and continuity with respect to μ

Suppose we have two periodic solutions of (1.10), w and $\bar{w}$, corresponding to μ and $\bar{\mu}$, such that $\| w \|_{B_0}$, $\| \bar{w} \|_{B_0} \leqq \theta$ and $| \mu - \mu_0 |$, $| \bar{\mu} - \mu_0 | < \theta$; $w, \bar{w} \in \bar{\mathrm{S}}$. We shall use the abbreviations

$$g(u_0 + w, s, t, \mu) = g(w, \mu) \quad \text{and} \quad \Phi(u_0 + w, s, t, \mu) = \Phi(w, \mu).$$

Then, if we write down (1.10) with w replaced by $\bar{w}$, and μ by $\bar{\mu}$, if we subtract the resulting equation from the unmodified (1.10), and if we carry out certain other obvious algebraic manipulations, we find that

$$\text{(4.1)} \qquad \frac{\partial(w - \bar{w})}{\partial t} - A(s, t)(w - \bar{w}) + \frac{\partial(w - \bar{w})}{\partial s} g(w, \mu) = \varphi(w, \bar{w}, \mu, \bar{\mu})$$

where

$$\text{(4.2)} \qquad \varphi(w, \bar{w}, \mu, \bar{\mu}) = \Phi(w, \mu) - \Phi(\bar{w}, \bar{\mu}) + \frac{\partial \bar{w}}{\partial s} (g(\bar{w}, \bar{\mu}) - g(w, \mu)).$$

Since Φ vanishes to the second order in w and $(\mu - \mu_0)^{1/2}$ and since g is of class C', we could find a positive number P such that

$$\text{(4.3)} \qquad \| \varphi(w, \bar{w}, \mu, \bar{\mu}) \|_{B_0} \leqq P\theta \| w - \bar{w} \|_{B_0} + P \,| \mu - \bar{\mu} |,$$

if we only knew that $\partial \bar{w}/\partial s$ were bounded by θ. Actually (1.10) may be satisfied only in a generalized sense so that $\partial \bar{w}/\partial s$ may not even exist. However, our solutions w and $\bar{w}$ both belong to the closure of the set $(\| w \|_{B0} < \theta) \cap (\| w \|_{B1} < \theta)$, and so, even if $\partial w/\partial s$ and $\partial \bar{w}/\partial s$ did not exist, we could find solutions of (1.10) w^* and $\bar{w}^*$ whose derivatives did exist and would be bounded by θ and such that

$$\text{(4.4)} \qquad \| w - w^* \|_{B_0} < \sigma, \qquad \| \bar{w} - \bar{w}^* \|_{B_0} < \sigma,$$

where σ is any preassigned positive number.

Hence we may at least assume that (4.1) and (4.3) are satisfied when w and $\bar{w}$ are replaced by w^* and $\bar{w}^*$ respectively. This approximation may be effected by modifying slightly the initial values $\alpha(s) = w(s, 0)$ and $\bar{\alpha}(s) = \bar{w}(s, 0)$, replacing them by slightly different but smoother functions $\alpha^*(s)$ and $\bar{\alpha}^*(s)$. But, of course, in doing this, we may sacrifice periodicity. The periodicity condition for the solution of (4.1), (cf. derivation of (3.7)), which can be written $T_u(\alpha - \bar{\alpha}) = H_u \varphi(w, \bar{w}, \mu, \bar{\mu})$, is replaced by the following inequality, which is a consequence of the fact that w^* and $\bar{w}^*$ approximate uniformly two solutions which were periodic from the outset:

$$\text{(4.5)} \qquad \| \xi \|_{\bar{B}_0} < 4\sigma,$$

where

$$\text{(4.6)} \qquad \xi = T_{u^*}(\alpha^* - \bar{\alpha}^*) - H_{u^*} \varphi(w^*, \bar{w}^*, \mu, \bar{\mu}),$$

and where $u^* = u_0 + w^*$. From the Theorem of Section 1, or the compact summarization embodied in (3.6), we have

$$w^* - \bar{w}^* = \dot{L}_{u^*}(\alpha^* - \bar{\alpha}^*) + K_{u^*}\varphi(w^*, \bar{w}^*, \mu, \bar{\mu}).$$

Hence, from (3.9) and (4.3)

$$\| w^* - \bar{w}^* \|_{B_0} \leqq C \| \alpha^* - \bar{\alpha}^* \|_{\bar{B}_0} + \theta CP \| w^* - \bar{w}^* \|_{B_0} + CP \mid \mu - \bar{\mu} \mid \tag{4.7}$$

From (4.6), (3.9), (3.10), (4.3), and (4.5), we have

$$\begin{aligned} \| \alpha^* - \bar{\alpha}^* \|_{\bar{B}_0} &= \| T_{u^*}^{-1} H_{u^*}\varphi(w^*, \bar{w}^*, \mu, \bar{\mu}) + T_{u^*}^{-1}\xi \|_{\bar{B}_0} \\ &\leqq \| T_{u^*}^{-1} H_{u^*}\varphi(w^*, \bar{w}^*, \mu, \bar{\mu}) \|_{\bar{B}_0} + \| T_{u^*}^{-1}\xi \|_{\bar{B}_0} \\ &\leqq C^2\theta P \| w^* - \bar{w}^* \|_{B0} + C^2 P \cdot \mid \mu - \bar{\mu} \mid + 4C\sigma. \end{aligned}$$

Combining this result with (4.7), we find that

$$\| w^* - \bar{w}^* \|_{B_0} \leqq (C^3\theta P + C\theta P) \| w^* - \bar{w}^* \|_{B_0} + (C^3 P + CP) \mid \mu - \bar{\mu} \mid + 4C^2\sigma.$$

We now choose θ so small that $C^3\theta P + C\theta P < \frac{1}{2}$. This is possible since both C and P are independent of θ, at least so long as $\theta < \delta$, as we have been assuming all along. Hence $\| w^* - \bar{w}^* \|_{B_0} < 2(C^3 P + CP) \mid \mu - \bar{\mu} \mid + 8C^2\sigma$, while from (4.4) we obtain

$$\begin{aligned} \| w - \bar{w} \|_{B_0} &\leqq \| w^* - \bar{w}^* \|_{B_0} + \| w - w^* \|_{B_0} + \| \bar{w}^* - \bar{w} \|_{B_0} \\ &< 2(C^3 P + CP) \mid \mu - \bar{\mu} \mid + (8C^2 + 2)\sigma. \end{aligned}$$

But σ can be taken arbitrarily small, while $\| w - \bar{w} \|_{B_0}$ is independent of σ. Hence $\| w - \bar{w} \|_{B_0} \leqq 2(C^3 P + CP) \cdot \mid \mu - \bar{\mu} \mid$. This proves simultaneously that for a given μ the solution is unique and that, considered as a function of μ, it satisfies a Lipschitz condition in μ and, hence, a fortiori, it is continuous, at least if the θ of the Theorem of Section 3 is taken sufficiently small.

The above technique can also be used to obtain the existence theorem of the previous section by a method of successive approximations, without using the Schauder fixed point theorem. As in so many other instances of the application of the Schauder theorem, the latter turns out to be a convenience rather than an essentiality.

Concluding comment

In this paper we have obtained periodic manifolds defined by continuous (Lipschitzian) functions, but not necessarily differentiable. It is highly probable, however, that we could have obtained manifolds of class C^q, if we had suitably sharpened our hypotheses on the smoothness of u_0, f, and g, using also, instead of the spaces B_0 and B_1, the spaces B_q and B_{q+1}, where B_p is defined as the space of functions of class C^p from $\mathfrak{M}$ to $\mathfrak{R}$ with $\| w \|_{B_p}$ defined as the maximum of the absolute values of the derivatives of the w's with respect to the s's and t of all orders from 0 to p, inclusive.

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