ON LOCAL HOMEOMORPHISMS OF EUCLIDEAN SPACES*

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Introduction

Consider the system of (real) non-linear differential equations

$$(1) \qquad x' = \Gamma x + F(x), \qquad \text{where} \quad F(x) = o(|\, x \,|) \quad \text{as} \quad x \to 0,$$

x is an N-vector, $x' = dx/dt$, Γ a constant N by N matrix, and $F(x)$ an N-vector valued function of class C^1 for small $|\, x \,|$. Let the eigenvalues $\gamma_1, \cdots, \gamma_N$ of Γ satisfy

$$(2) \qquad \operatorname{Re} \gamma_k \neq 0 \qquad \text{for } k = 1, \cdots, N.$$

This paper is concerned with the existence of C^1 maps

$$(3) \qquad u = x + \varphi(x), \qquad \text{where } \varphi(x) = o(|\, x \,|) \text{ as } x \to 0$$

defined for small $|\, x \,|$ and transforming (1) into the linear system

$$(4) \qquad R{:}u' = \Gamma u$$

for small $|\, u \,|$.

In the analytic case, this problem has been considered by Poincaré, Birkhoff, Siegel and others. When F is analytic and an analytic φ is sought, a comparison of coefficients leads to the diophantine inequalities

$$(5) \qquad \gamma_j \neq \sum_{k=1}^{N} n_k \gamma_k \qquad \text{for } j = 1, \cdots, N,$$

where $n_1, \cdots, n_N$ are non-negative integers with a sum $n_1 + \cdots + n_N > 1$.

In the case of a "contraction",

$$(6) \qquad \operatorname{Re} \gamma_k < 0 \qquad \text{for } k = 1, \cdots, N$$

(or, equivalently, a dilation $\operatorname{Re} \gamma_k > 0$) and where Γ has simple elementary divisors, Poincaré ([7], pp. xcix–cv), has shown that (5) is sufficient for the existence of an analytic, linearizing map (3) (when F is analytic). For another proof, see Sternberg ([9]). By an example which is, essentially,

$$(7) \qquad x' = -x, \qquad y' = -2y + x^2,$$

where a and y are scalars, Sternberg in [9], p. 812, shows that if (5) is violated, then there need not exist a linearizing map (3) of class C^2 (even though F is analytic).

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The general, non-analytic case has been treated by Sternberg ([9], [10]) and Nagumo and Isé ([6]). The latter prove, for example, that there exists a positive number r (depending only on Re $\gamma_1, \cdots$, Re γ_N) with the properties that if (5) holds for all sets of non-negative integers $n_1, \cdots, n_N$ subject to

$$1 < n_1 + \cdots + n_N < r$$

and if F is of class C^1 and each component F_j of F has partial derivatives of the form: polynomial $+ O(|x|^r)$ as $x \to 0$, then there exists a linearizing map (3) of class C^1.

The question to be considered below is whether or not there exists a linearizing map (3) of class C^1 when $F(x)$ is assumed to be, say, of class C^2, but *no diophantine inequalities* (5) *are assumed.* It is shown that the answer is in the affirmative for contractions and for binary ($N = 2$) systems. (This case $N = 2$ can be generalized somewhat to the case when the positive numbers in the set Re γ_1, $\cdots$, Re γ_N are nearly equal and the negative numbers in the set are nearly equal.) These results cannot be extended without further restrictions on the $\gamma_1, \cdots, \gamma_N$. For it is shown below that the analytic system of three (scalar) equations

$$x' = \alpha x, \qquad y' = (\alpha - \gamma)y + xz, \qquad z' = -\gamma z, \tag{8}$$

where $\alpha > \gamma > 0$, does not admit a linearizing map (3) of class C^1 (for small $|x|, |y|, |z|$). In fact, there does not exist a local map (3) of class C^1 with non-vanishing Jacobian which carries trajectories of (8) into those of

$$u' = \alpha u, \qquad v' = (\alpha - \gamma)v, \qquad w' = -\gamma w. \tag{9}$$

This answers a question raised by M. M. Peixoto.

Let $x = \xi(t, x_0)$ be the solution of (1) determined by the initial condition $\xi(0, x_0) = x_0$. Then

$$T^t : x^t = \xi(t, x)$$

is a "group" of transformations of a neighborhood of $x = 0$ into a neighborhood of $x^t = 0$. Also,

$$\xi(t, x) = e^{\Gamma t}x + F(t, x),$$

where $F(t, x) = o(|x|)$ as $x \to 0$. The problem of linearizing (1) is equivalent to the problem of finding a map R, say (3), such that $RT^t R^{-1}$ is linear $u^t = e^{\Gamma t}u$, that is, which linearizes T^t for every t. According to a lemma of Sternberg [9], p. 817, it is sufficient to find an R which linearizes T^1.

Correspondingly, instead of (1), the considerations below will deal with a local map

$$T: x^1 = Ax + X(x), \qquad \text{where } X(x) = o(|x|) \text{ as } x \to 0, \tag{10}$$

of a vicinity of $x = 0$ in x-space into a vicinity of $x^1 = 0$ in x^1-space. The question is the existence of a map (3) such that

$$RTR^{-1}\colon u^1 = Au. \tag{11}$$

Treatments of the analytic case of this problem when $N = 1$ go back to Abel, Schröder, Koenig (cf. de Bruijn [1]) and for contractions with arbitrary N to Leau; cf. [4] for references. (The early references were supplied to me by Professor L. Markus.)

An advantage to treating (10), rather than (1), is that the results can be used to study a system of differential equations in the vicinity of a periodic solution as well as in the vicinity of a stationary point.

Part 1. Contractions

1. Main theorem. Let x, x^1, u, u^1, X, φ denote N-vectors and A a (real) constant, N by N matrix.

(I) *Let A be a constant matrix with eigenvalues $a_1, \cdots, a_N$ satisfying*

$$0 < |a_j| < 1. \tag{1.1}$$

Let $X = X(x)$ be a function of class C^1 for small $|x|$ satisfying $X = \partial X/\partial x_1 = \cdots = \partial X/\partial x_N = 0$ at $x = 0$ and having partial derivatives which are uniformly Lipschitz continuous. Then, for the map $T\colon x \rightarrow x^1$,

$$T\colon x^1 = Ax + X(x), \tag{1.2}$$

there exists a map

$$R\colon u = x + \varphi(x), \tag{1.3}$$

of class C^1 for small x satisfying $\varphi = \partial\varphi/\partial x_1 = \cdots = \partial\varphi/\partial x_N = 0$ at $x = 0$ such that RTR^{-1} has the form

$$RTR^{-1}\colon u^1 = Au \tag{1.4}$$

for small $|u|$.

The uniform Lipschitz condition on the partial derivatives of X can be replaced by a uniform Hölder condition of order α, where $\alpha > \alpha_0$ and α_0 is a number depending only on $|a_1|, \cdots, |a_N|$ and satisfying $0 \leqq \alpha_0 < 1$. It will remain undecided whether or not the uniform Lipschitz (or suitable Hölder) condition can be replaced by a Lipschitz (or suitable Hölder) condition at $x = 0$ (as suggested by the conditions of Nagumo and Isé).

From the proof of (I), it will be seen that φ in (1.3) has uniformly Hölder continuous first derivatives (with a Hölder order depending only on $|a_1|, \cdots, |a_N|$).

2. The induction. The following notation will be used below: For a rectangular matrix Q, let the norm $|Q|$ be defined by max $|Q\xi|$ for $|\xi| = 1$, where ξ is a vector of appropriate dimension and $|\xi|$, $|Q\xi|$ denote Euclidean lengths in the corresponding spaces.

If x, y are vectors and $W = W(x, y)$ is a vector function, let $\partial_x W = (\partial W_p/\partial x_q)$, $\partial_y W = (\partial W_p/\partial y_q)$ denote the indicated (rectangular) Jacobian matrices.

For the proof of (I), it will be convenient to change notation. Write the N-vector x as (x, y, z), where x is an I-vector, y a J-vector, z a K-vector and $I + J + K = N$. Let A, B, C be square matrices of order I, J, K and with eigenvalues $a_1, \cdots, a_I, b_1, \cdots, b_J, c_1, \cdots, c_K$, respectively.

Induction hypothesis. In the local map

$$T: x^1 = Ax + X(x, y, z), \quad y^1 = By + Y(x, y, z), \quad z^1 = Cz, \tag{2.1}$$

let the eigenvalues of A, B, C satisfy

$$\begin{aligned} 0 < |a_1| \leqq \cdots \leqq |a_I| < |b_1| = \cdots \\ = |b_J| < |c_1| \leqq \cdots \leqq |c_K| < 1; \end{aligned} \tag{2.2}$$

let X, Y satisfy

(i) X, Y are defined, of class C^1 for small $|x|$, $|y|$, $|z|$ and $X, Y = o(|x| + |y| + |z|)$ as $(x, y, z) \to 0$;

(ii) the Jacobian matrices $\partial_x X$, $\partial_y X$ and $\partial_x Y$, $\partial_y Y$ are uniformly Lipshitz continuous with respect to (x, y, z);

(iii) The Jacobian matrices $\partial_z X$, $\partial_z Y$ are uniformly Lipschitz continuous with respect to (x, y); finally

(iv) The Jacobian matrices $\partial_z X$, $\partial_z Y$ are uniformly Hölder continuous with respect to z.

It is clear that (I) will be proved if the following is verified.

Induction assertion. There exists a local map R of the form

$$R: u = x - x(z), \quad v = y - y(x, y, z), \quad w = z, \tag{2.3}$$

where $x(z)$, $y(x, y, z)$ are of class C^1 for small $|x|$, $|y|$, $|z|$, are $o(|z|)$, $o(|x| + |y| + |z|)$ as z, $(x, y, z) \to 0$, respectively, and R is such that RTR^{-1} has the form

$$RTR^{-1}: u^1 = Au + U(u, v, w), \quad v^1 = Bv, \quad w^1 = Cw, \tag{2.4}$$

where

(i) $U(u, v, w)$ is of class C^1 for small $|u|$, $|v|$, $|w|$ and is $o(|u| + |v| + |w|)$ as $(u, v, w) \to 0$;

(ii) the Jacobian matrix $\partial_u U$ is uniformly Lipschitz continuous with respect to (u, v, w);

(iii) the Jacobian matrices $\partial_v U$, $\partial_w U$ are uniformly Lipschitz continuous with respect to u; finally,

(iv) the Jacobian matrices $\partial_v U$, $\partial_w U$ are uniformly Hölder continuous with respect to (v, w).

The "first" step of the usual induction proof can be omitted for "dummy" variables z, z^1 can be added to the map (2.1), if such variables are not present.

3. An invariant manifold. The induction assertion wll be proved in two stages. The first depends on the existence of a certain invariant manifold.

LEMMA. *Assume the Induction Hypothesis for the local map* (2.1). *Then there exists a manifold M invariant under T of the form*

$$M\colon x = x(z), \quad y = y(z), \tag{3.1}$$

where $x(z)$, $y(z)$ are of class C^1 for small $|z|$, are $o(|z|)$ as $z \to 0$, and have uniformly Hölder continuous partial derivatives.

As to the existence of an invariant M of class C^1, the assumptions (ii)–(iv) of the induction hypothesis can be replaced by the assumption that the partial derivatives of X, Y satisfy a Hölder condition in (x, y, z) only at the point $x = y = z = 0$. In which case, the partial derivatives of $x(z)$, $y(z)$ satisfy a Hölder condition at $z = 0$. If (iv) is suitably modified in both the induction hypothesis and assertion, then these properties of M would suffice for the proof of (I). (But, of course, the assumptions (ii), (iii) concerning Lipschitz continuity will be used elsewhere below.)

PROOF OF THE LEMMA. In view of the symmetry of the assumptions and assertions in x and y, it will be convenient to change the notation again so that (x, y), (X, Y), (A, B), (a, b) become x, X, A, a respectively. Thus, (2.1) becomes

$$T\colon x^1 = Ax + X(x, z), \quad z^1 = Cz, \tag{3.2}$$

and the assumptions on X are the same as in the induction hypothesis (and X does not depend on y). A manifold

$$M\colon x = x(z) \tag{3.3}$$

is invariant under (3.2) if and only if $x(z)$ satisfies the functional equation $x^1 = x(z^1)$, that is,

$$Ax(z) + X(x(z), z) = x(Cz) \tag{3.4}$$

or, equivalently,

$$x(z) = Ax(C^{-1}z) + X(x(C^{-1}z), C^{-1}z). \tag{3.5}$$

After preliminary (separate) linear changes of the independent variables x and z, it can be supposed that there exist numbers a, c, d satisfying

$$0 < a < c < d < 1 \tag{3.6}$$

and

$$|A| \leqq a, \qquad |C| \leqq d, \qquad |C^{-1}| \leqq 1/c. \tag{3.7}$$

The number $\epsilon > 0$ will be fixed below. Let

$$E_0 = \{z\colon |z| \leqq \epsilon\} \quad \text{and} \quad E_m = \{Cz\colon z \in E_{m-1}\} \qquad \text{for } m = 1, 2, \cdots; \tag{3.8}$$

so that $E_m = C(E_{m-1})$. Since $|Cz| \leqq d|z|$ and $d < 1$, it is clear that the ellipsoid E_m is a proper subset of E_{m-1} and that E_m shrinks to the origin as $m \to \infty$.

Define $x(z)$ to be 0 in a thin shell $\epsilon - \eta \leqq |z| \leqq \epsilon + \eta$ containing $|z| = \epsilon$. Then (3.5) defines $x(z)$ in a thin shell containing the set $C(|z| = \epsilon)$ as a function of class C^1 with partial derivatives which are uniformly Hölder continuous. The domain of $x(z)$ can be extended (arbitrarily) so as to include $E_0 - E_1$ in such a way that $x(z)$ remains of class C^1 and $\partial_z x(z)$ is uniformly Hölder continuous. Then (3.5) defines $x(z)$ successively on $E_1 - E_2$, $E_2 - E_3, \cdots$; so that $x(z)$ is of class C^1 on $0 < |z| \leqq \epsilon$. Finally, put $x(0) = 0$.

(1) *Estimates for* $|x(z)|$. Let L, α be positive constants such that

$$|X(x, z)| \leqq L(|x|^{1+\alpha} + |z|^{1+\alpha}). \tag{3.9}$$

Choose δ so that

$$0 < \delta \leqq \alpha \quad \text{and} \quad a/c^{1+\delta} < 1, \tag{3.10}$$

$\epsilon > 0$ so small that

$$\theta \equiv a/c^{1+\delta} + L\,\epsilon^{\alpha(1+\delta)}/c^{(1+\alpha)(1+\delta)} < 1, \tag{3.11}$$

and L_1 so as to satisfy

$$(1 - \theta)L_1 \geqq L\,\epsilon^{\alpha-\delta}/c^{1+\delta} \tag{3.12}$$

and, on $E_0 - E_1$,

$$|x(z)| \leqq L_1|z|^{1+\delta}. \tag{3.13}$$

It will be verified that (3.13) holds for $|z| \leqq \epsilon$. Assume that (3.13) holds on $E_{m-1} - E_m$ for some $m \geqq 1$. Let $x \in E_m - E_{m+1}$, so that $C^{-1}z \in E_{m-1} - E_m$. By (3.5), (3.9),

$$|x(z)| \leqq |Ax(C^{-1}z)| + L(|x(C^{-1}z)|^{1+\alpha} + |C^{-1}z|^{1+\alpha}),$$

and so, by (3.7) and the assumption of (3.13) on $E_{m-1} - E_m$,

$$|x(z)| \leqq aL_1(|z|/c)^{1+\delta} + L[L_1(|z|/c)^{(1+\alpha)(1+\delta)} + (|z|/c)^{1+\alpha}].$$

Since $|z| \leqq \epsilon$, this can be written as

$$|x(z)| \leqq L_1|z|^{1+\delta}\theta + L|z|^{1+\delta}\,\epsilon^{\alpha-\delta}/c^{1+\delta},$$

by virtue of (3.11). Hence (3.12) implies (3.13).

(2) *Estimates for* $|\partial_z x(z)|$. By assumption, there exist positive constants L, α satisfying

$$\begin{aligned} &|\partial_x X(x, z)| \leqq L(|x| + |z|), \\ &\qquad\qquad |\partial_z X(x, z)| \leqq L(|x| + |z|^\alpha). \end{aligned} \tag{3.14}$$

Let δ satisfy (3.10), so that (3.13) holds for $|z| \leqq \epsilon$. Choose $\epsilon > 0$ so small that

$$\theta \equiv a/c^{1+\delta} + L[L_1(\epsilon/c)^{1+\delta} + \epsilon/c]/c^{1+\delta} < 1. \tag{3.15}$$

Let L_2 be chosen so that

$$(3.16)\qquad (1-\theta)L_2 \geqq L[L_1\,\epsilon/c^{1+\delta} + \epsilon^{\alpha-\delta}/c^{\alpha}]/c$$

and that, on $E_0 - E_1$,

$$(3.17)\qquad |\,\partial_z x(z)\,| \leqq L_2|\,z\,|^{\delta}.$$

It will be verified that (3.17) holds for $|\,z\,| \leqq \epsilon$. By (3.5),

$$\partial_z x(z) = \{A(\partial_z x^0) + (\partial_x X^0)(\partial_z x^0) + (\partial_z X^0)\}C^{-1},$$

where the superscript 0 on x, X means that the argument is $C^{-1}z$, $(x(C^{-1}(z)),$ $C^{-1}z)$, respectively. Since the norm of a matrix product Q_1Q_2 satisfies $|\,Q_1Q_2\,| \leqq |\,Q_1\,|\cdot|\,Q_2\,|$, (3.7) implies that

$$|\,\partial_z x\,| \leqq \{a + |\,\partial_x X^0\,|\}|\,\partial_z x^0\,|/c + |\,\partial_z X^0\,|/c.$$

Thus, by (3.14),

$$|\,\partial_z x\,| \leqq \{a + L(|\,x^0\,| + |\,z\,|/c)\}|\,\partial_z x^0\,|/c + L\{|\,x^0\,| + (|\,z\,|/c)^{\alpha}\}/c.$$

Assume (3.17) for $z \in E_0 - E_{m-1}$. Then, for $z \in E_1 - E_m$, (3.13) implies that

$$|\,\partial_z x\,| \leqq \{a + L(L_1(|\,z\,|/c)^{1+\delta} + |\,z\,|/c)\}L_2|\,z\,|^{\delta}/c^{1+\delta}$$
$$+ L\{L_1(|\,z\,|/c)^{1+\delta} + (|\,z\,|/c)^{\alpha}\}/c.$$

This gives

$$|\,\partial_z x\,| \leqq L_2|\,z\,|^{\delta}\theta + |\,z\,|^{\delta}\,L\{L_1\,\epsilon/c^{1+\delta} + \epsilon^{\alpha-\delta}/c^{\alpha}\}/c,$$

since $|\,z\,| \leqq \epsilon$ and θ is given by (3.15). Hence (3.16) implies (3.17) on $E_1 - E_m$. Consequently, (3.17) holds for $0 < |\,z\,| \leqq \epsilon$.

It follows that $\partial_z x(z)$ exists and is 0 at $z = 0$.

(3) *Holder continuity of* $\partial_z x$. The argument just used to prove (3.3) can be modified to show that there exists a constant L_3 satisfying

$$(3.18)\qquad |\,\partial_z x(z+\Delta z) - \partial_z x(z)| \leqq L_3\,|\,\Delta z\,|^{\delta}$$

for z, $z + \Delta z \in E_{m-1} - E_m$ and $m = 1, 2, \cdots$. Also, by (3.17),

$$|\,\partial_z x(z+\Delta z) - \partial_z x(z)| \leqq L_2(|\,z\,|^{\delta} + |\,z+\Delta z\,|^{\delta}).$$

In view of (3.7) and (3.8), it is seen that $|\,z\,|, |\,z+\Delta z\,| \leqq d^{m-1}\epsilon$. Hence if $0 < \eta < 1$, the last two formula lines give

$$(3.19)\qquad |\,\partial_z x(z+\Delta z) - \partial_z x(z)| \leqq L_4\theta^m\,|\,\Delta z\,|^{\delta\eta}$$

if $L_4 = (2L_2\epsilon/d)^{1-\eta}\,L_3^{\eta}$ and $\theta = d^{1-\eta}$. By continuity, (3.19) is valid if z, $z + \Delta z$ are in the closure of $E_m - E_{m-1}$.

Consequently, for arbitrary $|\,z\,|, |\,z+\Delta z\,| \leqq \epsilon$,

$$|\,\partial_z x(z+\Delta z) - \partial_z x(z)| \leqq L_4(\sum_{m=1}^{\infty}\theta^m)|\,\Delta z\,|^{\delta\eta};$$

that is, (3.18) is valid for arbitrary z, $z + \Delta z$ in $|z| \leqq \epsilon$ if L_3 is replaced by $L_4\theta/(1 - \theta)$ and δ by $\delta\eta$. This completes the proof of the Lemma.

4. Preliminary change of variables. Consider a map (3.2) as in the proof of the Lemma and a change of variables R of the form

$$(4.1) \qquad R: u = x - x(z), \quad w = z; \qquad R^{-1}: x = u + x(w), \quad z = w.$$

A simple calculation shows that RTR^{-1} is of the form

$$u^1 = Au + Ax(w) + X(u + x(w), w) - x(Cw), \quad w^1 = Cw.$$

If $x = x(z)$ is an invariant manifold M, then (3.4) shows that

$$(4.2) \qquad RTR^{-1}: u^1 = Au + X^*(u, w), \quad w^1 = Cw,$$

where

$$(4.3) \qquad X^*(u, w) = X(u + x(w), w) - X(x(w), w).$$

Since $\partial_x X$ is uniformly Lipschitz continuous in (x, z), it follows that $\partial_u X^*$ is uniformly Lipschitz continuous in (u, w). Also,

$$\partial_w X^* = [\partial_x X(u + x(w), w) - \partial_x X(x(w), w)]\partial_z x(w) + [\partial_z X(u + x(w), w) - \partial_z X(x(w), w)]$$

shows that $\partial_w X^*$ is uniformly Lipschitz continuous in u and uniformly Hölder continuous in w. Furthermore,

$$(4.4) \qquad |\partial_u X^*| \leqq \text{Const.}(|u| + |w|), \qquad |\partial_w X^*| \leqq \text{Const.}\,|u|.$$

Since $X^*(0, w) = 0$ by (4.3), the first relation in (4.4) implies that

$$(4.5) \qquad |X^*(u, w)| \leqq \text{Const.}(|u| + |w|)|u|.$$

If $\partial_w X^*$ is Hölder continuous of order δ, then the last part of (4.4) gives

$$(4.6) \qquad |\partial_w X^*(u, w + \Delta w) - \partial_w X^*(u, w)| \leqq \text{Const.}\,|u|^\eta\,|\Delta w|^{\delta(1-\eta)}$$

for all η, $0 \leqq \eta \leqq 1$.

5. Proof of the Induction Assertion. If u, v are renamed (x, y), z, respectively, it is seen that the assumption (i) in the Induction Hypothesis can be supplemented by

$$(5.1) \qquad |V(x, y, z)| \leqq L(|x| + |y| + |z|)(|x| + |y|),$$

where $V = X, Y$ (cf. (4.5)); and (iv) by

$$(5.2) \qquad |\partial_z V(x, y, z + \Delta z) - \partial_z V(x, y, z)| \leqq L(|x| + |y|)^\eta\,|\Delta z|^{\delta(1-\eta)}$$

for some $\delta > 0$ and all η, $0 \leqq \eta \leqq 1$ (cf. (4.6)) and by

$$(5.3) \qquad |\partial_z V(x, y, z)| \leqq L(|x| + |y|)$$

(cf. the second part of (4.4)).

It will be shown that there is a local map R

$$(5.4)\qquad \begin{aligned} R&: u = x, \quad v = y - \varphi(x, y, z), \quad w = z, \\ R^{-1}&: x = u, \quad y = v + \psi(u, v, w), \quad z = w, \end{aligned}$$

transforming (2.1) into the form (2.4) with the desired properties.

Note that φ, ψ in (5.4) satisfy

$$(5.5)\qquad \varphi(x, y, z) \equiv \psi(u, v, w).$$

A simple calculation shows that (2.1) and (5.4) give

$$\begin{aligned} u^1 &= Au + X(u, v + \psi, w), \\ RTR^{-1}: v^1 &= B(v + \psi) + Y(u, v + \psi, w) \\ &\quad - \varphi(Au + X, B(v + \psi) + Y, Cw), \\ w^1 &= Cw, \end{aligned}$$

where the argument of ψ is (u, v, w) and that of X, Y in $\varphi(Au + X, \cdots)$ is $(u, v + \psi, w)$. Thus, the relations

$$(5.6)\qquad RTR^{-1}: u^1 = Au + X(u, v + \psi(u, v, w), w), \quad v^1 = Bv, w^1 = Cw,$$

hold if φ, ψ satisfy

$$B\psi = \varphi(Au + X, B(v + \psi) + Y, Cw) - Y(u, v + \psi, w).$$

By (5.4) and (5.5), this means that

$$(5.7)\qquad B\varphi(x, y, z) = \varphi(Ax + X, By + Y, Cz) - Y,$$

where the argument of X and Y is (x, y, z).

The functional equation (5.7) for φ will be solved by successive approximations. It can be supposed, after suitable (separate) linear transformations of the variables x, y, z, that the matrices A, B, C satisfy

$$(5.8)\qquad |A| \leqq a, \qquad |B| \leqq b^1, \qquad |B^{-1}| \leqq 1/b^2, \qquad |C| \leqq c$$

where a, b^1, b^2, c are constants satisfying

$$(5.9)\qquad a < b^1 < c < 1 \quad \text{and} \quad b^1c/b^2 < 1.$$

Define a sequence of successive approximations φ_0, φ_1, $\cdots$ as follows:

$$\varphi_0(x, y, z) \equiv 0, \qquad \varphi_1(x, y, z) = -B^{-1}Y(x, y, z)$$

and, for $n \geqq 1$,

$$(5.10)\qquad B\varphi_n(x, y, z) = \varphi_{n-1}(Ax + X, By + Y, Cz) - Y(x, y, z).$$

Put

$$(5.11)\qquad \varphi^n = \varphi_n - \varphi_{n-1} \qquad \text{for} \quad n = 1, 2, \cdots,$$

so that

$$\varphi^1(x, y, z) = -B^{-1}Y(x, y, z) \tag{5.12}$$

and

$$\varphi^n(x, y, z) = B^{-1}\varphi^{n-1}(Ax + X, By + Y, Cz) \qquad \text{for} \quad n > 1. \tag{5.13}$$

Since A, B, C are contractions, it is clear that if $\epsilon > 0$ is sufficiently small, the relations (5.10) define functions φ_n of class C^1 for $|x|, |y|, |z| \leqq \epsilon$.

It will first be shown that there exist positive constants L_1, θ such that $\theta < 1$ and

$$|\varphi^n(x, y, z)| \leqq L_1\theta^{n-1}(|x| + |y| + |z|)(|x| + |y|). \tag{5.14}$$

In view of (5.1) and (5.12), L_1 can be chosen so as to satisfy (5.14) for $n = 1$ and $|x|, |y|, |z| \leqq \epsilon$.

Note that $|Ax + X| \leqq a|x| + L(|x| + |y| + |z|)(|x| + |y|)$ and an analogous relation is valid for $|By + Y|$. Hence

$$|Ax + X| + |By + Y| + |Cz| \leqq (a + r)|x| \\ + (b^1 + r)|y| + c|z| \leqq c(|x| + |y| + |z|),$$

where $r > 0$ is arbitrarily small if $|x|, |y|, |z| \leqq \epsilon$ and ϵ is sufficiently small. Also,

$$|Ax + X| + |By + Y| \leqq (a + r)|x| \\ + (b^1 + r)|y| \leqq (b^1 + r)(|x| + |y|).$$

If (5.14) holds when n is replaced by $n - 1$, then (5.13) shows that

$$|\varphi^n(x, y, z)| \leqq L_1\theta^{n-2}[c(b^1 + r)/b^2]\,[|x| + |y| + |z|]\,[|x| + |y|].$$

This implies (5.14) if $\epsilon > 0$ is so small that

$$\theta \equiv c(b^1 + r)/b^2 < 1; \tag{5.15}$$

(cf. (5.8)).

It will now be shown that if ϵ, L_1 and θ are appropriately chosen (with $L_1 > 0$, $0 < \theta < 1$), then

$$|\partial_x\varphi^n|, \ |\partial_y\varphi^n| \leqq L_1\theta^{n-1}(|x| + |y| + |z|). \tag{5.16}$$

In order to see this, note that

$$\partial_x\varphi^n = B^{-1}[(\partial_x\varphi^{n-1,0})(A + \partial_x X) + (\partial_y\varphi^{n-1,0})(\partial_x Y)],$$

$$\partial_y\varphi^n = B^{-1}[(\partial_x\varphi^{n-1,0})(\partial_y X) + (\partial_y\varphi^{n-1,0})(B + \partial_y Y)],$$

where the superscript 0 signifies that the argument of φ^{n-1} is $(Ax + X, By + Y, Cz)$. If (5.16) is assumed when n is replaced by $n - 1$, then

$$| \partial_x\varphi^n | \leqq L_1(\theta_1^{n-2}/b^2)c(| x | + | y | + | z |)(a + r),$$

$$| \partial_y\varphi^n | \leqq L_1(\theta_1^{n-2}/b^2)c(| x | + | y | + | z |)(b^1 + r).$$

Thus, (5.16) follows if (5.15) holds.

In the same way, it is shown that

$$(5.17) \qquad | \Delta\partial_x\varphi^n |, | \Delta\partial_y\varphi^n | \leqq L_1\theta^{n-1}(| \Delta x | + | \Delta y | + | \Delta z |),$$

if, e.g., $\Delta\partial_x\varphi^n = \partial_x\varphi^n(x + \Delta x, y + \Delta y, z + \Delta z) - \partial_x\varphi^n(x, y, z)$.

Next, it will be verified that

$$(5.18) \qquad | \partial_z\varphi^n(x, y, z)| \leqq L_1\theta^{n-1}(| x | + | y |).$$

In fact,

$$(5.19) \quad \partial_z\varphi^n = B^{-1}[(\partial_x\varphi^{n-1,0})(\partial_z X) + (\partial_y\varphi^{n-1,0})(\partial_z Y) + (\partial_z\varphi^{n-1,0})C].$$

If (5.18) is assumed when n is replaced by $n - 1$, then (5.16) and (5.3) show that

$$| \partial_z\varphi^n | \leqq L_1(\theta^{n-2}/b^2)(b^1 + r)(| x | + | y |)(c + r)$$

and (5.18) holds for n if $\theta = (b^1 + r)(c + r)/b^2$. If $\epsilon > 0$ is sufficiently small, then $\theta < 1$.

In the same way, it is shown that

$$(5.20) \quad | \partial_z\varphi^n(x + \Delta x, y + \Delta y, z) - \partial_z\varphi^n(x, y, z)| \leqq L_1\theta^{n-1}(| \Delta x | + | \Delta y |).$$

Finally, it will be shown that for all $\eta (<1)$ near 1,

$$(5.21) \quad | \partial_z\varphi^n(x, y, z + \Delta z) - \partial_z\varphi^n(x, y, z)| \leqq L_1\theta^{n-1}(| x | + | y |)^{\eta} | \Delta z |^{(1-\eta)\delta}.$$

If (5.21) holds when n is replaced by $n - 1$, then (5.19) shows that the left side of (5.21) is majorized by

$$L_1(\theta^{n-2}/b^2)[(b^1 + r)(| x | + | y |)]^{\eta}(c + r)[(c + r)| \Delta z |]^{(1-\eta)\delta}.$$

This gives (5.21) if $\theta = (b^1 + r)^{\eta}(c + r)^{1+(1-\eta)\delta}/b^2$. Also, if $\epsilon > \theta$, $1 - \eta > 0$ are sufficiently small, then $\theta < 1$.

It follows from (5.14), (5.16), (5.17), (5.19), (5.20), (5.21) that the functional equation (5.7) has a solution φ of class C^1 on a set $| x |, | y |, | z | \leqq \epsilon$ satisfying $| \varphi | \leqq L_1(| x | + | y | + | z |)(| x | + | y |)$, $\partial_x\varphi$, $\partial_y\varphi$ are uniformly Lipschitz continuous in (x, y, z); finally, $\partial_z\varphi$ satisfies $| \partial_z\varphi | \leqq L_1(| x | + | y |)$, is uniformly Lipschitz continuous in (x, y) and uniformly Hölder continuous in z.

The map (5.4) changes T into (5.6). If (2.4) is identified with (5.6) it follows that

$$U(u, v, w) \equiv X(x, y, z) \text{ by virtue of } (5.4).$$

It remains to check (i)–(iv) of the Induction Assertion. First, (i) is obvious. In order to verify (ii), note that $\partial_u U = \partial_x X + (\partial_y X)(\partial_u \psi)$. Also, by (5.5) $\partial_u \psi = \partial_x \varphi + (\partial_y \varphi)(\partial_u \psi)$, so that $\partial_u \psi = (I - \partial_y \varphi)^{-1}(\partial_x \varphi)$. It is then clear that $\partial_u U$ is uniformly Lipschitz continuous in (x, y, z), hence in (u, v, w). Thus (ii) holds.

Note that $\partial_v U = (\partial_y X)(I + \partial_v \psi)$ and that $\partial_v \psi = (I - \partial_y \varphi)^{-1}(\partial_y \varphi)$. Hence $\partial_v U$ is uniformly Lipschitz continuous in (u, v, w). This proves the parts of (iii), (iv) concerning $\partial_v U$.

Similarly, $\partial_w U = (\partial_y X)(\partial_w \psi) + \partial_z X$ and $\partial_w \psi = (I - \partial_y \varphi)^{-1}\partial_z \varphi$. Hence, $\partial_w U$ is uniformly Lipschitz continuous in (x, y) and uniformly Hölder continuous in z. Since changes Δu, Δv, Δw in u, v, w produce changes Δx, Δy, Δz in x, y, z satisfying $\Delta u = \Delta x$, $| \Delta y | \leqq$ Const. $(| \Delta u | + | \Delta v | + | \Delta w |)$, $\Delta w = \Delta z$, it follows that $\partial_w U$ is uniformly Lipschitz in (u, v) and uniformly Hölder continuous in w. This gives that parts of (iii), (iv) concerning $\partial_w U$ and completes the proof of the induction and of (I).

Part 2. A theorem on the general case

6. Statement of results. If (I) is combined with a procedure of Sternberg ([10], pp. 627–628), one obtains the following:

(II) *In the map*

$$T: x^1 = Ax + X(x, z), \quad z^1 = Cx + Z(x, z), \tag{6.1}$$

let x be an I-vector and z a K-vector; A, C constant square matrices with eigenvalues $a_1, \cdots, a_I, c_1, \cdots, c_K$ satisfying

$$0 < | a_1 | \leqq \cdots \leqq | a_I | < 1 < | c_1 | \leqq \cdots \leqq | c_K | ; \tag{6.2}$$

X, Z are functions of class C^1 for small $| x |$ and $| z |$, are $o(| x | + | z |)$ as $(x, z) \to 0$ and have uniformly Lipschitz continuous partial derivatives. Then there exists a map

$$R: u = x - \varphi(x, z), \quad w = z - \chi(x, z) \tag{6.3}$$

of class C^1 for small $| x |$, $| z |$ in which

$$\varphi, \chi = o(| x | + | z |) \qquad \textit{as} \quad (x, z) \to 0, \tag{6.4}$$

φ, χ possess uniformly Hölder continuous partial derivatives and (6.3) transforms (6.1) into the form

$$RTR^{-1}: u^1 = Au + U(u, w), \quad w^1 = Cw + W(u, w), \tag{6.5}$$

where

$$U(0, w) = W(u, 0) = 0 \tag{6.6}$$

and

$$U(u, 0) = W(0, w) = 0. \tag{6.7}$$

The relations (6.6) mean that the manifolds $u = 0$ and $w = 0$ are invariant under (6.5). The relations (6.7) mean that (6.5) is a linear map on each of these invariant manifolds.

The proof of (II) depends on a variant of the following theorem. This theorem is stated and its proof is indicated in only a few lines by Sternberg ([9], Theorem 9, pp. 822–823) for $m \geqq 2$.

(III) *In the map* T *given by* (6.1), *let* x, z, A, C *be as in* (II). *Let* X, Z *be defined for small* $|x|$ *and* $|z|$, *satisfy* $X, Z = o(|x| + |z|)$ *as* $(x, z) \to 0$, *and be of class* C^m, $m \geqq 1$ (*or analytic*). *Then there exists a map* (6.3) *of class* C^m (*or analytic*) *for small* $|x|$ *and* $|z|$, *satisfying* (6.4) *and such that if* RTR^{-1} *is given by* (6.5), *then* U, W *satisfy* (6.6).

For the sake of completeness, a proof of (III) will be given below. This proof will make the following remark clear:

Remark. In (III), replace the assumption that X, Z is of class C^m, $m \geqq 1$ (or analytic) by either (α) X, Z is uniformly Lipschitz continuous with a Lipschitz constant $c = c(\epsilon)$ on $|x|, |z| \leqq \epsilon$ satisfying $c(\epsilon) \to 0$ as $\epsilon \to 0$ or (β) X, Z is of class C^m, $m \geqq 1$, with uniformly Lipschitz continuous mth order partial derivatives. Then (III) remains valid if the assertion that (6.3) is of class C^m (or analytic) is replaced by the analogue of (α) or (β) in which φ, χ replace X, Z.

When x, z are scalars, (III) is a theorem of Poincaré ([7], pp. 202–204) in the analytic case and of Hadamard ([3]) in the C^1 case. The binary case of (III) and the Remark is given by Sternberg in [8]. The analogue of (III) for autonomous differential systems, in the analytic case, is due to Liapounoff ([5], p. 291); for the non-analytic case, cf. Coddington and Levinson ([2], p. 333). The results on differential systems can be deduced from (III) and the Remark.

(II) follows by first applying the case $m = 1$ of (β) in the Remark, then applying (I) first to RTR^{-1} on the manifold $w = 0$ and then to the inverse of the resulting map on the manifold $u = 0$.

7. Proof of (III). We first seek a map R of the form

$$(7.1) \qquad R: u = x, \quad w = z - \chi(x); \qquad R^{-1}: x = u, \quad z = w + \chi(u),$$

such that RTR^{-1} is given by (6.4) with

$$(7.2) \qquad W(u, 0) = 0.$$

From (6.1) and (7.1), RTR^{-1} is given by

$$u^1 = Au + X(u, w + \chi)$$
$$w^1 = Cw + C\chi + Z(u, w + \chi) - \chi(Au + X(u, w + \chi)),$$

where the argument of χ is u unless otherwise indicated. Thus (7.2) is equivalent to the functional equation

$$(7.3) \qquad C\chi(u) + Z(u, \chi(u)) = \chi(Au + X(u, \chi(u)))$$

for χ.

After preliminary (separate) linear transformations of the variables x and z, it can be supposed that the matrices A, C satisfy

$$(7.4) \qquad |A| = a < 1, \qquad |C^{-1}| = 1/c < 1.$$

The functional equation (7.3) will be solved by the successive approximations $\chi_0(u) \equiv 0$, $\chi_1(u) = -C^{-1}Z(u, 0)$ and, more generally, for $n \geqq 1$,

$$(7.5) \qquad \chi_n(u) = C^{-1}\{\chi_{n-1}(Au + X(u, \chi_{n-1}(u)) - Z(u, \chi_{n-1}(u))\}.$$

(The existence of χ_n on some sphere $|u| \leqq \epsilon$, where $\epsilon > 0$ does not depend on n, will follow from the considerations below.)

(It is clear that if $X, Z \in C^m$, $m \geqq 2$, then the successive approximations and their partial derivatives of order $\leqq m - 1$ are uniformly convergent for small $|u|$. But the difficulty in appraising the mth order partials of χ_n is that there is nothing like a Lipschitz condition available on the mth order partials of X, Z.)

It will first be shown that if $\epsilon > 0$ is sufficiently small, then

$$(7.6) \qquad |\chi_n(u)| \leqq \epsilon \qquad \text{for} \quad |u| \leqq \epsilon$$

and $n = 0, 1, \cdots$. To this end, note that the assumptions on X, Z and A show that if $\epsilon > 0$ is sufficiently small, then

$$(7.7) \qquad |Z(x, w)| \leqq (c-1)\epsilon, \qquad |Ax + X(x, w)| \leqq \epsilon \quad \text{if} \quad |x|, |w| \leqq \epsilon.$$

Assume that (7.6) holds if n is replaced by $n - 1$, then (7.4), (7.5) and (7.7) show that, for $|u| \leqq \epsilon$, $|\chi_n(u)| \leqq (1/c)[\epsilon + (c-1)\epsilon] \leqq \epsilon$. This proves (7.6) for $n = 0, 1, \cdots$.

It is clear that χ_n is of class C^1 (since X, Z are). Its Jacobian matrix $\partial\chi_n$ is given by

$$(7.8) \qquad \partial\chi_n(u) = C^{-1}\{(\partial\chi_{n-1}^0)[A + \partial_x X^0 + (\partial_z X^0)(\partial\chi_{n-1})] - [\partial_x Z^0 + (\partial_z Z^0)(\partial\chi_{n-1})]\},$$

where the superscript 0 on χ_{n-1} and on X, Z indicates that their arguments are $Au + X^0$ and $(u, \chi_{n-1}(u))$, respectively.

It will be shown that if $\epsilon > 0$ is sufficiently small, then there exists a $\delta_\epsilon > 0$ such that

$$(7.9) \qquad |\partial\chi_n(u)| \leqq \epsilon \qquad \text{if} \quad |u| \leqq \delta_\epsilon$$

for $n = 0, 1, \cdots$. To this end, note that if $\eta > 0$, then there exists a $\delta = \delta(\eta) > 0$ such that

$$(7.10) \qquad |\partial_x X|, |\partial_z X|, |\partial_x Z|, |\partial_z Z| \leqq \eta \qquad \text{if} \quad |x|, |z| \leqq \delta(\eta) \leqq \eta.$$

Choose ϵ and $\eta = \eta(\epsilon)$ so that

$$\eta \leqq \epsilon, \qquad \eta < (c-a)\epsilon, \qquad a + \eta/\epsilon + 2\eta + \eta\epsilon \leqq c.$$

Let $\delta_\epsilon = \delta(\eta(\epsilon))$. Assume (7.9) if n is replaced by $n - 1$. Then (7.4), (7.6), (7.7), (7.8), (7.10) show that

$$|\partial\chi_n(u)| \leqq (1/c)[\epsilon(a + \eta + \eta\epsilon) + \eta + \eta\epsilon] \leqq \epsilon,$$

so that (7.9) holds for $n = 0, 1, \cdots$.

It will now be shown that the Jacobian matrices $\partial\chi_0, \partial\chi_1, \cdots$ are equicontinuous, that is, that there exists a function $h(\delta)$ defined for small $\delta > 0$ satisfying $h(\delta) \to 0$ as $\delta \to 0$ and

$$(7.11) \qquad |\partial\chi_n(u) - \partial\chi_n(u')| \leqq h(\delta) \qquad \text{if} \quad |u - u'| \leqq \delta$$

for $n = 0, 1, \cdots$.

For any function $g = g(u)$, $g = g(x, z)$, etc., let $\Delta g = g(u') - g(u)$, $\Delta g = g(x', z') - g(x, z)$, etc. Let $h_1(\delta)$ be defined for small $\delta > 0$, satisfy $h_1(\delta) \to 0$ as $\delta \to 0$ and

$$(7.12) \qquad |\Delta\partial_x X|, |\Delta\partial_z X|, |\Delta\partial_x Z|, |\Delta\partial_z Z| \leqq h_1(\delta)$$

for $|x - x'|, |z - z'| \leqq \delta$. Note that if $|x|, |x'|, |z|, |z'|$ are sufficiently small, then

$$(7.13) \qquad \begin{aligned} |\Delta Z| &\leqq \tfrac{1}{2}|x - x'| + \tfrac{1}{2}|z - z'|, \\ |\Delta(Ax + X(x, z))| &\leqq \tfrac{1}{2}(a + 1)|x - x'| + \tfrac{1}{2}(1 - a)|z - z'|. \end{aligned}$$

Since $\chi_{n-1}(u)$ is uniformly Lipschitz continuous with an arbitrarily small Lipschitz constant on $|u| \leqq \epsilon$ for sufficiently small ϵ (cf. (7.9)), the last part of (7.13) implies that

$$(7.14) \qquad |\Delta(Au + X(u, \chi_{n-1}(u))| \leqq |u - u'|.$$

Using the analogue of $\Delta(g_1(u)g_2(u)) = g_1(u')\Delta g_2 + g_2(u)\Delta g_1$, it follows that if (7.11) holds when n is replaced by $n - 1$ and if $|u|, |u'| \leqq \delta_\epsilon$ and

$$|u - u'| \leqq \delta,$$

then $|\Delta\partial\chi_n|$ is majorized by $1/c$ times the sum of $h(\delta)[a + \eta + \eta\epsilon]$, $\epsilon[h_1(\delta) + \eta h(\delta) + h_1(\delta)\epsilon]$, $h_1(\delta)$ and $\eta h(\delta) + h_1(\delta)\epsilon$. This gives (7.11) if ϵ, η are so small small that $c > a + 2\eta + 2\eta\epsilon$ and

$$h(\delta) \geqq h_1(\delta)(1 + 2\epsilon + \epsilon^2)/(c - a - 2\eta - 2\eta\epsilon).$$

It will now be verified that the sequence $\chi_0, \chi_1, \cdots$ is uniformly convergent for small $|u|$; in fact, it will be shown that

$$(7.15) \qquad |\chi_n(u) - \chi_{n-1}(u)| \leqq Lr^n, \quad 0 < r < 1,$$

for some constant L and $n = 1, 2, \cdots$.

The difference $|\chi_n(Au + X(u, \chi_n(u)) - \chi_{n-1}(Au + X(u, \chi_{n-1}(u))|$ is majorized by the sum of $|\chi_n(Au + X(u, \chi_n(u))) - \chi_{n-1}(Au + X(u, \chi_n(u)))|$ and $|\chi_{n-1}(Au + X(u, \chi_n(u))) - \chi_{n-1}(Au + X(u, \chi_{n-1}(u)))|$. If (7.15) holds, the first of these differences is at most Lr^n. An appraisal for the second difference follows from the fact that χ_{n-1} and X are uniformly Lipschitz continuous with the respective Lipschitz constants ϵ and η if $|u|$ and $|x|, |z|$ do not exceed $\delta = \delta_\epsilon$. Thus the second difference is at most $\epsilon\eta \,|\chi_n(u) - \chi_{n-1}(u)| \leqq L\epsilon r^n$. Then (7.5) shows that

$$|\chi_{n+1}(u) - \chi_n(u)| \leqq (1/c)\{Lr^n + L\epsilon\eta r^n + \eta L r^n\}$$

which is at most Lr^{n+1} if $r = (1 + \epsilon\eta + \eta)/c$. Also $r < 1$ if ϵ, η are small enough. Hence (7.15) holds for $n = 2, 3, \cdots$ if L is chosen so that it holds for $n = 1$.

This proves that $\chi_0, \chi_1, \cdots$ converge uniformly to a (continuous) solution χ of (7.3) satisfying $\chi(u) = o(|u|)$ as $u \to 0$; cf. (7.9). Since $\partial\chi_0, \partial\chi_1, \cdots$ are uniformly bounded and equicontinuous, a subsequence is uniformly convergent. This implies that χ is of class C^1.

A similar proof shows that if X, Z are analytic, so is χ. Also, if $X, Z \in C^m$, then the sequences of partial derivatives of χ_n of order $\leqq m$ can be shown to be uniformly bounded and equicontinuous, so that $\chi \in C^m$.

(III) now follows by applying the procedure just completed for T to the inverse of RTR^{-1} in order to obtain a map for which the analogue of $U(0, w) = 0$ holds.

Part 3. A special case

8. Cases related to $N = 2$. The object of this part of the paper is to prove the possibility of linearizing a binary ($N = 2$) map which is not a contraction. A somewhat more general result will be proved. In order to state this result, the following notation will be used: Let A be a (real) N by N matrix with eigenvalues $a_1, \cdots, a_N$ satisfying $0 < |a_j| < 1$. According to (I) and the remarks following it, there exists a number $\lambda = \lambda(|a_1|, \cdots, |a_N|)$, $0 < \lambda \leqq 1$, with the property that if $T: x^1 = Ax + X(x)$ is a local map in which $X(x) = o(|x|)$ as $x \to 0$ and $X(x)$ has uniformly Lipschitz continuous partial derivatives, then there is a C^1 local map $R: u = x + \varphi(x)$ linearizing T, $\varphi(u) = o(|u|)$ as $u \to 0$ and $\varphi(u)$ has partial derivatives which are uniformly Hölder continuous of order λ.

The arguments in Sternberg [9] or in the last part of the proof of (I) show, for example, that $\lambda(|a_1|, \cdots, |a_N|) = 1$ if $|a_N|^2 < |a_1|$.

(IV) *Let the map T, given by* (6.1), *be as in* (II). *Let*

$$\lambda = \lambda(|a_1|, \cdots, |a_I|), \qquad \mu = \lambda(1/|c_K|, \cdots, 1/|c_1|)$$

and let the eigenvalues of A, C satisfy

$$(8.1) \qquad |a_I|^\lambda |c_K|/|c_1| < 1 \quad \textit{and} \quad |c_1|^\mu |a_1|/|a_I| > 1.$$

Then there exists a C^1 map R of the form (6.3) *which satisfies* (6.4) *and linearizes T*,

$$(8.2) \qquad RTR^{-1}: u^1 = Au, \quad w^1 = Cw.$$

The condition (8.1) is redundant if, for example,

$$(8.3) \qquad |a_1| = \cdots = |a_I| \quad \text{and} \quad |c_1| = \cdots = |c_K|.$$

In particular, (8.1) holds if x, z are scalars ($I = K = 1$ and $N = 2$). Actually, in the latter case, the smoothness assumptions on X, Z can be relaxed somewhat.

The proof of (IV) can be modified to show that R can be chosen so that φ, χ have uniformly Hölder continuous partial derivatives.

9. Proof of (IV). In view of (III), it can be supposed that

$$X(0, z) = Z(x, 0) = 0; \tag{9.1}$$

hence

$$|\partial_z X(x, z)| \leqq L|x| \quad \text{and} \quad |\partial_x Z(x, z)| \leqq L|z|. \tag{9.2}$$

By the proof of (II), there is a local map of the form $R: u = x - \varphi_1(x)$, $w = z - \varphi_2(z)$ and $R^{-1}: x = u + \psi_1(u)$, $z = w + \psi_2(w)$, such that φ_1, ψ_1 and φ_2, ψ_2 have uniformly Hölder continuous partial derivatives vanishing at $x = z = 0$ of order λ, μ, respectively, and such that if RTR^{-1} is given by (6.5), then (6.6) and (6.7) hold. It is readily verified that

$$U = A\psi_1 + X(u + \psi_1, w + \psi_2) - \varphi_1(Au + A\psi_1 + X(u + \psi_1, w + \psi_2)),$$

$$W = C\psi_2 + Z(u + \psi_1, w + \psi_2) - \varphi_2(Cu + C\psi_2 + Z(u + \psi_1, w + \psi_2)),$$

where the argument of ψ_1, ψ_2 is u, w, respectively. The conditions (6.7) show that

$$U = [X(u + \psi_1, w + \psi_2) - \varphi_1(Au + A\psi_1 + X(u + \psi_1, w + \psi_2))]_{w=0}^{w=w},$$

$$W = [Z(u + \psi_1, w + \psi_2) - \varphi_2(Cw + C\psi_2 + Z(u + \psi_1, w + \psi_2))]_{u=0}^{u=u}.$$

It follows from (9.2) that

$$|\partial_w U| \leqq L|u| \quad \text{and} \quad |\partial_u W| \leqq L|w|;$$

also the uniform Lipschitz continuity of the partials of X, Z and the uniform Hölder continuity of those of φ_1, φ_2, ψ_1, ψ_2 show that

$$|\partial_u U| \leqq L|w|^{\mu} \quad \text{and} \quad |\partial_w W| \leqq L|u|^{\lambda}.$$

Let u, w, RTR^{-1} be renamed x, z, T, then it can be supposed that X, Z are of class C^1 and that (9.1), (9.2), and

$$X(x, 0) = Z(z, 0) = 0, \tag{9.3}$$

$$|\partial_x X| \leqq L|z|^{\mu}, \qquad |\partial_z Z| \leqq L|x|^{\lambda} \tag{9.4}$$

hold; finally, (9.2) and (9.3) give

$$|X(x, z)| \leqq L|x||z| \quad \text{and} \quad |Z(x, z)| \leqq L|x||z|. \tag{9.5}$$

The inverse map T^{-1} is of the form

$$T^{-1}: x = A^{-1}x^1 + X_1(x^1, z^1), \quad z = C^{-1}z^1 + Z_1(x^1, z^1) \tag{9.6}$$

where X_1, Z_1 satisfy the analogues (9.1)–(9.5). Also, by virtue of (6.1) or (9.6),

$$-X(x, z) \equiv AX_1(x^1, z^1), \quad -Z(x, z) = CZ_1(x^1, z^1). \tag{9.7}$$

The condition (8.1) implies that if the variables x, z are subjected to separate, suitable, linear transformations, then it can be assumed that the matrices A, C satisfy

$$(9.8) \qquad |A|^{\lambda}|C^{-1}|\;|C| < 1 \quad \text{and} \quad |C^{-1}|^{\mu}|A^{-1}|\;|A| < 1.$$

Consider a map R of the form (6.3). Then (8.2) holds if and only if χ satisfies the functional equation

$$(9.9) \qquad C\chi(x, z) = \chi(Ax + X(x, z), Cx + Z(x, z)) - Z(x, z)$$

and φ satisfies an analogous equation. In view of (6.1) and the first part of (9.7), the latter can be written in terms of (x^1, z^1)-variables as

$$(9.10) A^{-1}\varphi(x^1, z^1)$$
$$= \varphi(A^{-1}x^1 + X_1(x^1, z^1), C^{-1}z^1 + Z_1(x^1, z^1)) - X_1(x^1, z^1).$$

Since the relation (9.9), (9.10) are completely analogous, only (9.10) will be considered and x, z, X, Z will be written in place of x^1, z^1, X_1, Z_1.

Without loss of generality, it can be supposed that X, Z are defined for all x, z (and not only small $|x|$, $|z|$), satisfy (9.1)–(9.5), and $X \equiv Z \equiv 0$ if $|x|^2 + |z|^2 \geqq r^2$, where $r > 0$ is a preassigned number. If this is not the case, replace X, Z by $X(x, z)\lambda(|x|^2 + |z|^2)$, $Z(x, z)\lambda(|x|^2 + |z|^2)$, where $\lambda(\rho)$ is a smooth function of the single variable ρ satisfying $\lambda(\rho) \equiv 1$ for $0 \leqq \rho \leqq \frac{1}{2}r^2$, $\lambda(\rho) \equiv 0$ for $\rho > r^2$ and $0 \leqq \lambda \leqq 1$ for all ρ.

Note that this procedure leaves a factor L in (9.2), (9.4), (9.5) which does not depend on r. This is clear for the factor in (9.5). In (9.4), consider, for example, $\partial_x(\lambda X) = (\partial_x\lambda)X + \lambda(\partial_x X)$. But λ can be chosen so that

$$|d\lambda(\rho)/d\rho| \leqq 3/r^2$$

and so, $|\partial_x\lambda(|x|^2 + |z|^2)| \leqq 6/r$. Also $|X| \leqq L|x||z| \leqq Lr|z|^{\mu}$ for $|x| \leqq r$. Hence $|\partial_x(\lambda X)| \leqq 7L|z|^{\mu}$.

The inequalities (9.2) and (9.4) show that the partial derivatives of X, Y have bounds of the form Lr, Lr^{λ} or Lr^{μ}. Thus, if r is sufficiently small, T is one-to-one for all (x, z).

Since X, Z are defined for all (x, z), it is possible to define a sequence of successive approximations by $\varphi_0(x, z) \equiv 0$, $\varphi_1(x, z) = -AX(x, z)$ and, for $n \geqq 1$,

$$(9.11) \quad \varphi_n(x, z) = A\{\varphi_{n-1}(A^{-1}x + X(x, z), C^{-1}z + Z(x, z)) - X(x, z)\}.$$

Let $\varphi^n = \varphi_n - \varphi_{n-1}$ for $n = 1, 2, \cdots$, so that $\varphi^1 = -AX(x, z)$ and

$$(9.12) \qquad \varphi^n(x, z) = A\varphi^{n-1}(A^{-1}x + X(x, z), C^{-1}z + Z(x, z))$$

for $n = 2, 3, \cdots$.

If $r > 0$ above is sufficiently small, there exist positive constants $\theta(<1)$, L_1 such that

$$(9.13) \quad |\varphi^n| \leqq L_1\theta^n|x||z|, \qquad |\partial_x\varphi^n| \leqq L_1\theta^n|z|^{\mu}, \qquad |\partial_z\varphi^n| \leqq L_1\theta^n|x|$$

for $n = 1, 2, \cdots$ and all (x, z). For, if the first inequality in (9.13) is assumed when n is replaced by $n - 1$, then (9.12) implies that

$$| \varphi^n(x, z) | \leqq L_1\theta^{n-1}| A |\cdot| A^{-1}x + X(x, z) |\cdot| C^{-1}z + Z(x, z) |.$$

But by (9.5), $| X(x, z) | \leqq Lr| x |$ and $| Z(x, z) | \leqq Lr| z |$ since $X = Z = 0$ if $| x | > r$ or $| z | > r$. Thus

$$| \varphi^n(x, z) | \leqq L_1\theta^{n-1}| A |(| A^{-1} | + Lr)(| C^{-1} | + Lr)| x || z |.$$

By (9.8), it can be supposed that $r > 0$ is fixed so small that

$$\theta \equiv | A |(| A^{-1} | + Lr)(| C^{-1} | + Lr) < 1.$$

This gives the first part of (9.13) for $n = 2, 3, \cdots$ if it holds for $n = 1$. The other parts of (9.13) are proved similarly.

The inequalities (9.13) imply the uniform convergence of the successive approximations (9.11) on compact (x, z)-sets to a C^1-solution of (9.10).

Part 4. Counter-example

10. The example. The object of this part of the paper is to show that (I) is false if the restriction (1.1) that the linear part of (I) is a contraction is replaced merely by $0 < | a_j | \neq 1$, even if $X(x)$ is analytic. The example T arises from the system of three scalar differential equations

$$(10.1) \qquad x' = \alpha x, \quad y' = (\alpha - \gamma)y + \epsilon xz, \quad z' = -\gamma z,$$

where $\epsilon \neq 0$ and $\alpha > \gamma > 0$. The solution of this system starting at (x, y, z) for $t = 0$ is

$$(10.2) \qquad x(t) = xe^{\alpha t}, \quad y(t) = (y + \epsilon xzt)e^{(\alpha-\gamma)t}, \quad z(t) = ze^{-\gamma t}.$$

If $t = 1$, the map $T:x \rightarrow x(1)$ is

$$(10.3) \qquad T:x^1 = ax, \quad y^1 = ac(y + \epsilon xz), \quad z^1 = cz,$$

where $a > 1 > c > 0$ (in fact, $a = e^{\alpha}$, $c = e^{-\gamma}$) and $\epsilon \neq 0$.

It will be shown that there is no map R of class C^1 of the type described in (I) which linearizes (10.3); hence, none which linearizes (10.1). Actually it will be shown that there is no map R of the type specified such that if

$$(10.4) \qquad RTR^{-1}:u^1 = U(u, v, w), \quad v^1 = V(u, v, w), \quad w^1 = W(u, v, w),$$

then

$$(10.5) \qquad V(u, 0, w) \equiv 0.$$

In particular, there is no such map R of class C^1 which transforms (10.1) into the diagonal form

$$(10.6) \quad u' = \lambda(u, v, w)\alpha u, \quad v' = \lambda(u, v, w)(\alpha - \gamma)v, \quad w' = -\lambda(u, v, w)\gamma w,$$

where $\lambda(u, v, w)$ is a continuous function and

$$(10.7) \qquad \lambda(0, 0, 0) = 1.$$

In this connection, there is, of course, no loss of generality in supposing that the linear part of R is the identity map.

The negative assertion concerning (10.6) means that there is no C^1 map (with non-vanishing Jacobian) mapping a neighborhood of $(x, y, z) = 0$ into a vicinity of $(u, v, w) = 0$ and transforming trajectories of (10.1) into those of (10.6).

11. The functional equations. Suppose that the map R given by

(11.1) $$R^{-1}: x = u + f(u, v, w), \quad y = v + g(u, v, w), \quad z = w + h(u, v, w)$$

transforms (10.3) into (10.4). Let $Q = RTR^{-1}$, so that $TR^{-1} = R^{-1}Q$. Then

(11.2) $$a(u + f) = U + f(U, V, W), \quad c(w + h) = W + h(U, V, W),$$

(11.3) $$ac[v + g + \epsilon(u + f)(w + h)] = V + g(U, V, W),$$

where the argument of f, g, h on the left is (u, v, w).

Suppose that (10.5) holds and let $v = 0$ in (11.2), (11.3). In order to shorten notation, write $j(u, w)$ in place of $j(u, 0, w)$ for $j = f, g, h, U, V, W$. Then

(11.4) $$a(u + f(u, w)) = U + f(U, W), \quad c(w + h(u, w)) = W + h(U, W),$$

and

(11.5) $$ac[g(u, w) + \epsilon(u + f(u, w)(w + h(u, w))] = g(U, W).$$

The C^1 map $(u, w) \to (x, z)$ given by

(11.6) $$x = u + f(u, w), \qquad z = w + h(u, w)$$

has a (local) C^1 inverse, say

(11.7) $$u = p(x, z), \qquad w = q(x, z).$$

Then (11.4) implies, for $U = U(u, w)$, $W = W(u, w)$,

(11.8) $$U = p(ax, cz), \qquad W = q(ax, cz).$$

Substitute (11.6), (11.7), (11.8) into (11.5), after introducing the abbreviation

(11.9 $$e(x, z) \equiv g(u, w) = g(p(x, z), q(x, z)),$$

to obtain

(11.10) $$ac[e(x, z) + \epsilon xz] = e(ax, cz).$$

It will be shown that this functional equation has no solution $e = e(x, z)$ of class C^1 for small $|x|$, $|z|$ satisfying

(11.11) $$e = e_x = e_z = 0 \qquad at \quad x = z = 0$$

if

(11.12) $$\epsilon \neq 0, \qquad a > 1 > c > 0.$$

12. Proof. Assume the existence of such an $e(x, z)$. Let n be a positive integer and put

$$e_j = e(x/a^j, c^{n-j}z) \qquad \text{for} \quad j = 0, 1, \cdots, n. \tag{12.1}$$

In view of (11.12), e_j is defined for small $|x|$, $|z|$ and $j = 0, \cdots, n$. The functional equation (11.10) gives

$$ace_j + \epsilon a^{-j+1}c^{n-j+1}xz = e_{j-1}. \tag{12.2}$$

This implies $ace_j = e_{j-1}$ if $x = 0$ or $z = 0$; hence, $(ac)^n e_n = e_0$ if $x = 0$ or $z = 0$. This means

$$(ac)^n e(x/a^n, 0) = e(x, 0), \qquad (ac)^n e(0, z) = e(0, c^n z). \tag{12.3}$$

The first of these relations and (11.11) show that $e(x, 0) = (ac)^n o(|x|/a^n) = o(1)$ as $n \to \infty$. This implies the first of the two relations

$$e(x, 0) \equiv 0 \quad \text{and} \quad e(0, z) \equiv 0. \tag{12.4}$$

The second is obtained similarly.

By the last part of (12.4), $e_n = e(x/a^n, z)$ satisfies

$$e(x/a^n, z) = e(x/a^n, z) - e(0, z) \sim (x/a^n)e_x(0, z), \qquad \text{as } n \to \infty,$$

uniformly for small $|x|$, $|z|$. This gives the first of the two assertions:

$$a^n e_n \text{ and } c^{-n}e_0 \text{ are uniformly bounded}, \qquad \text{as } n \to \infty, \tag{12.5}$$

for small $|x|$, $|z|$. The second follows similarly.

Multiply (12.2) by $a^{j-1}c^{-n+j-1}$ to obtain

$$a^j c^{-n+j} e_j + \epsilon xz = a^{j-1}c^{-n+j-1}e_{j-1}.$$

Adding these relations for $j = 1, \cdots, n$ gives

$$a^n e_n + n\epsilon xz = c^{-n}e_0. \tag{12.6}$$

But if $\epsilon xz \neq 0$, this contradicts (12.5) when $n \to \infty$.

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