EXISTENCE THEOREMS FOR PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL SYSTEMS

By Lamberto Cesari*

In the present paper I give the main idea underlying a recent approach to existence theorems for periodic solutions of nonlinear differential systems, and I summarize a number of applications and results. This approach has been developed by the author, J. K. Hale, R. A. Gambill, W. R. Fuller, *et al.* in a number of years (see Bibliography).

A presentation of some of the results in the analytical case has appeared in [4, §§4.5 and 8.5]. Here a new presentation is given in a more general setting. All applications and results under consideration belong to the class of the problems of perturbation of linear differential systems. I shall deal elsewhere with straightforward nonlinear differential systems as well as with perturbation problems of nonlinear differential systems.

§1. Schauder's fixed point theorem and perturbation problems

Let us consider the differential system

(1) $y' = Ay + \epsilon q(y, t, \epsilon), \quad y = (y_1, \cdots, y_n), \quad q = (q_1, \cdots, q_n)$

where ϵ is a small parameter, $|\epsilon| \leq \epsilon_0$, A a constant $n \times n$ matrix, and q is periodic in t of some period $T = 2\pi/\omega$, and L-integrable in $[0, T]$, (or alternatively q is independent of t and (1) is autonomous). For the sake of brevity we suppose $A = \text{diag}(\rho_1, \cdots, \rho_n)$ where the numbers ρ_j may depend on ϵ, are continuous functions of ϵ, and $\rho_j(0) = i\tau_j = ia_j\omega/b_j$, $a_j \gtreqless 0$, $b_j > 0$, integers. Suppose that

(2) $|q_j(x, t, \epsilon)| \leq K(t)$, $j = 1, \cdots, n$, for all $|y_s| \leq R$, $s = 1, \cdots, n$, $|\epsilon| \leq \epsilon_0$;

(3) There is a continuous monotone function $\zeta(\eta) \geq 0$, $\eta \geq 0$, $\zeta(0) = 0$, such that

$$|q_j(y^1, t, \epsilon^1) - q_j(y^2, t, \epsilon^2)| \leq \zeta(\eta)K(t)$$

for all $|y_s^1|, |y_s^2| \leq R$, $|\epsilon^1|, |\epsilon^2| \leq \epsilon_0$, $|y_s^1 - y_s^2| \leq \eta$, $|\epsilon^1 - \epsilon^2| \leq \eta$, $s = 1, \cdots, n$, where $K(t)$ is a fixed function L-integrable in $[0, T]$ (a constant if (1) is autonomous).

We may try to find a solution "close" to a solution of the linear system $z' = A(0)z$ of the form

$$z(t) = (c_1e^{i\tau_1 t}, \cdots, c_ne^{i\tau_n t}), \quad c_j \neq 0 \text{ constants.}$$

---

* This research was partially supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract Number AF 49(638)-382. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Let $B = A(0) = \operatorname{diag}(i\tau_1, \cdots, i\tau_n)$. Let Ω be the space of all continuous vector functions $\varphi(t) = (\varphi_1, \cdots, \varphi_n)$ of period $2\pi b_0/\omega$, $b_0 = b_1 b_2 \cdots b_n$, with $m[e^{i\tau_j t}\varphi_j(t)] = c_j$, $j = 1, \cdots, n$, where m denotes the usual mean value. Let us take in Ω a uniform topology. Let Ω_R be the subset of Ω consisting of all φ with $|\varphi_j| \leq R$, $j = 1, \cdots, n$. Let $\psi = \mathfrak{T}\varphi$ be the transformation of Ω_R into Ω defined by

$$(4) \qquad \psi(t) = z(t) + \epsilon e^{Bt} \int e^{-Bu}\{q[\varphi(u), u, \epsilon] - D[\varphi]\varphi(u)\}\, du,$$

where $D = \operatorname{diag}(d_1, \cdots, d_n)$ is defined by

$$(5) \qquad c_j d_j = m\{e^{-i\tau_j t} q_j[\varphi(t), t, \epsilon]\}, \quad j = 1, \cdots, n.$$

By this definition, for each component ψ_j, the integrand is periodic of mean value zero and the integral denotes the unique primitive which is periodic and of mean value zero. Finally, ψ belongs to Ω for φ belonging to Ω_R, i.e. $\mathfrak{T}\Omega_R \subset \Omega$.

If $|c_j| < R$, $j = 1, \cdots, n$, and $|\epsilon|$ sufficiently small there is a closed sphere Ω_0 about $z(t)$ in Ω_R such that $\mathfrak{T}\Omega_0 \subset \Omega_0$ [7]. Also, as shown in [7], there is a compact closed convex subset $\Omega_0^* \subset \Omega_0$ such that $\mathfrak{T}\Omega_0^* \subset \Omega_0^*$. Thus, by Schauder's fixed point theorem, there is at least one fixed element $y(t)$ in Ω_0^*, i.e., such that $\mathfrak{T}y = y$. The vector function $y(t)$ satisfies the integral equation

$$y(t) = z(t) + \epsilon e^{Bt} \int e^{-Bu}\{q[y(u), u, \epsilon] - D[y]y(u)\}\, du,$$

as well as the differential system

$$(6) \qquad y'(t) = (B - \epsilon D)y + \epsilon q(y, t, \epsilon).$$

If the relation (determining system)

$$(7) \quad B - \epsilon D = A, \quad \text{or} \quad ia\omega_j/b_j - \epsilon\, d_j(a, b, c, \omega, \epsilon) = \rho_j, \quad j = 1, \cdots, n,$$

is satisfied, then $y(t)$ is a periodic solution of the given system (1).

The analysis of the transformation $\mathfrak{T}$ and of the determining system (7) lead to actual existence theorems for periodic solutions and families of periodic solutions of the given differential system (1) (§§3–6).

Under a Lipschitz condition of the type

$$(8) \qquad |q_j(\overset{1}{y}, t, \epsilon) - q_j(\overset{2}{y}, t, \epsilon)| \leq K(t) \textstyle\sum_s |\overset{1}{y}_s - \overset{2}{y}_s|$$

the transformation $\mathfrak{T} \mid \Omega_0$ is a contraction and into [7]. Hence, not only is there a unique fixed point in Ω_0, but such a fixed point can be approached uniformly by the method of successive approximations

$$(9) \qquad y^{(0)}(t) = z(t), \quad y^{(m)}(t) = \mathfrak{T}y^{(m-1)}(t), \quad m = 1, 2, \cdots.$$

This method, together with some variants of it for the analytic case, has been studied by the author, J. K. Hale, R. A. Gambill, W. R. Fuller, et al. in a series of papers ([6], [9], [13], [15], [16], [25] and others in the bibliography. Its convergence, which was proved directly in a number of cases, is now a corollary ([7]) of the remark above under a Lipschitz condition (8).

Note that the "terms" subtracted in (4) have an interpretation. The so-called secular terms are all among them. More terms are subtracted so as to assure that the fixed element satisfies a system (6) of the same form of the original.

By application of this method, and various extensions of it, existence theorems for periodic solutions of periodic or autonomous systems and theorems for their stability have been proved.

§2. Some extensions and remarks

As a first extension of the approach mentioned in §1, I mention that we may consider [7] a system analogous to (1)

$$y' = Ay + \epsilon q(y, t, \epsilon), \quad A = \operatorname{diag}(A_1, A_2) = A(\epsilon), \tag{10}$$

where $y = (y_1, \cdots, y_n)$, $q = (q_1, \cdots, q_n)$, A is an $n \times n$-matrix whose elements are constants or continuous functions of ϵ, $A_1 = \operatorname{diag}(\rho_1, \cdots, \rho_\nu)$, $\rho_j(0) = i\tau_j = ia_j\omega/b_j$, $a_j \gtreqless 0$, $b_j > 0$ integers, $j = 1, \cdots, \nu$, and A_2 has characteristic roots, $\rho_j(\epsilon)$, $j = \nu + 1, \cdots, n$ (continuous functions of ϵ for $|\epsilon| \leq \epsilon_0$) with $\rho_j(0) \neq im\,\omega/b_0$, $b_0 = b_1b_2 \cdots b_\nu$, for all $j = \nu + 1, \cdots, n$, $m = 0, \pm1, \pm2, \cdots$. Of course, it is enough to know that A admits of a canonical form A_0 as above ([7]). In these conditions $z(t)$ and $D(t)$ of §1 are replaced by

$$z(t) = (c_1e^{i\tau_1 t}, \cdots, c_\nu e^{i\tau_\nu t}, 0, \cdots, 0), \quad c_j \text{ constants},$$

$$D(t) = (d_1, \cdots, d_\nu, 0, \cdots, 0),$$

and the corresponding determining system (7) becomes [7]

$$ia_j\omega/b_j - \epsilon\, d_j(a, b, c, \omega, \epsilon) = \rho_j, \quad j = 1, \cdots, \nu.$$

Note that the condition concerning A_2 is satisfied if either $\alpha_j(0) \neq 0$, or $\alpha_j(0) = 0$ and $\sigma_j(0) \neq im\,\omega/b_0$, $j = \nu + 1, \cdots, n$, $m = 0, \pm1, \pm2, \cdots$.

Instead of (10) we could consider ([7]) a system

$$y' = Ay + F(t) + q(y, t, \epsilon) \tag{11}$$

where $F(t) = (F_1, \cdots, F_n)$ is a periodic vector function of period $2\pi/\omega$ whose components are L-integrable in $[0, 2\pi/\omega]$ satisfying

$$\int_0^{2\pi/\omega} e^{i\tau_j t}F(t)\, dt = 0, \quad i = 1, \cdots, \nu.$$

Under these conditions the system

$$y' = By + F(t),$$

$B = \operatorname{diag}(B_1, A_2)$, $B_1 = \operatorname{diag}(i\tau_1, \cdots, i\tau_\nu)$, has a periodic solution $Y(t)$ and the transformation $y = Y(t) + z$ reduces (11) to a system (10).

Instead of (10) we could consider ([7]) a system

$$y' = Ay + q(y, t, \epsilon), \tag{12}$$

$y = (y_1, \cdots, y_n)$, $q = (q_1, \cdots, q_n)$, with A and q as in system (10) and $q(y, t, 0) = 0$.

Analogously, we could consider ([7]) a system

$$(13) \qquad y' = Ay + q(y, t, \epsilon) + g(y, t, \epsilon),$$

$y = (y_1, \cdots, y_n)$, $q = (q_1, \cdots, q_n)$, $g = (g_1, \cdots, g_n)$, with A and q as in system (12) and $|g_j(y, t, \epsilon)| \leq |y| \zeta(|y|)K(t)$, where $|y| = |y_1| + \cdots + |y_n|$, and $\zeta(\eta)$ as in §1. Analogously, we could suppose, for instance $g_j \equiv 0, j = 1, \cdots, \nu$, and

$$(14) \qquad g_j(y, t, \epsilon) | \leq |\bar{y}| \xi(|\bar{y}|)K(t), \quad j = \nu + 1, \cdots, n,$$

with $|\bar{y}| = |y_{\nu+1}| + \cdots + |y_n|$ (see §3).

The case of system (10) with $\nu = 0$ is rather trivial, since no characteristic root has real part zero for $\epsilon = 0$. There are on determining equations and the existence of a periodic solution is immediate. On the other hand, for $\nu = 0$, an extension in another direction is possible. Namely, we may suppose that the functions $q_j(y, t, \epsilon)$ are almost periodic in t for every y and E. More precisely we may suppose ([7]) that each $q_j(y, t, \epsilon)$ has a development

$$\sum c_s e^{i\lambda_s t}, \quad \sum |c_s|^2 \leq M,$$

where M can be taken independently of y and ϵ for $|y_j| \leq R$, $|\epsilon| \leq \epsilon_0$, $j = 1, \cdots, n$.

If we suppose that the elements of the matrix A and vector q are analytic functions of ϵ and $\epsilon, y_1, \cdots, y_n$, respectively, then Lipschitz condition (8) is satisfied and we may ask whether the series whose sums are the periodic functions $y_j(t)$

$$(15) \qquad y_j(t) = y_j^{(0)}(t) + \sum_{m=1}^{\infty} [y_j^{(m)}(t) - y_j^{(m-1)}(t)], \quad j = 1, \cdots, n,$$

are power series in ϵ, i.e., whether $y_j^{(0)}(t)$ is independent of ϵ, and each successive bracket contains only terms of degree $\leq m$ in ϵ. This is not the case even in particular situations (the functions q_j linear in $y_1, \cdots, y_n$ with coefficients independent of ϵ). J. K. Hale has modified the method of successive approximations (9) in such a way that each successive bracket in (15) contains only terms of degree $\leq m$.

The restriction $c_j \neq 0$ of §1 can be weakened (see [7] for a preliminary discussion).

3. First existence theorems for periodic solutions of real systems

Consider the real system of order $N = \mu + n$

$$(16) \qquad \begin{aligned} x_j'' + 2\alpha_j x_j' + \sigma_j^2 x_j &= \epsilon f_j(x, x', t, \epsilon), \quad j = 1, \cdots, \mu, \\ x_j' + \beta_j x_j &= \epsilon f_j(x, x', t, \epsilon), \quad j = \mu + 1, \cdots, n, \end{aligned}$$

where $-\infty < t < +\infty$, $(') = d/dt$, $0 \leq \mu \leq n$, $x = (x_1, \cdots, x_n)$, $x' = (x_1', \cdots, x_\mu')$, $\sigma_j > 0$, α_j, β_j real constants or continuous functions of ϵ, ϵ real,

$|\epsilon| \leq \epsilon_0$, and where $f_j(x, x', t, \epsilon)$, $j = 1, \cdots, n$, are real functions of the n-vector x, of the μ-vector x', of t and ϵ, for $|x_j| \leq R, j = 1, \cdots, n$, $|x_j'| \leq R$, $j = 1, \cdots, \nu$, $-\infty < t < +\infty$, $|\epsilon| \leq \epsilon_0$. We shall suppose that the functions f_j satisfy the same conditions of the q's in §1 with respect to the vector $(x_1, \cdots, x_n, x_1', \cdots, x_u')$. In particular we suppose that condition (8) is satisfied and the functions f_j are periodic of period $T = 2\pi/\omega$. We shall denote by ρ_{j1}, ρ_{j2} the roots of the equation $\rho^2 + 2\alpha_j\rho + \sigma_j^2 = 0$. By reordering equations and unknowns, if needed, we may suppose that for convenient integers ν, r, $0 \leq \nu \leq \mu \leq r \leq n$, we have $\alpha_j(0) = 0$, $\sigma_j(0) = \tau_j = a_j\omega/b_j$, $a_j > 0$, $b_j > 0$ integers, $j = 1, \cdots, \nu$, $\rho_{j1}(0)$, $\rho_{j2}(0) \neq im\,\omega/b_0$, $m = 0, 1, 2, \cdots$, $b_0 = b_1b_2 \cdots b_\nu$, $j = \nu + 1, \cdots, \mu$, $\beta_j(0) \neq 0$, $j = \mu + 1, \cdots, r$, $\beta_j(0) = 0$, $j = r + 1, \cdots, n$. Note that we have $\rho_{j1}(0) = i\tau_j$, $\rho_{j2}(0) = -i\tau_j$, $j = 1, \cdots, \nu$, and that, for $|\epsilon|$ sufficiently small we certainly have $\alpha_j^2(\epsilon) < \sigma_j^2(\epsilon)$, $j = 1, \cdots, \nu$, hence $\rho_{j1}, \rho_{j2} = -\alpha_j \pm i\gamma_j$, $\gamma_j > 0$, with $\gamma_j = (\sigma_j^2 - \alpha_j^2)^{\frac{1}{2}}$, $j = 1, \cdots, \nu$. The "zero characteristic roots" of (16) with $\epsilon = 0$ are those corresponding to the equations (16) with $j = r + 1, \cdots, n$. Note that the extreme cases above, i.e., $\nu = 0$, $\nu = \mu$, $r = \mu$, $r = n$, as well as $\mu = 0$, $\mu = n$ are not excluded. It will be convenient to suppose that the functions $\alpha_j(\epsilon)$, $\sigma_j(\epsilon)$, $j = 1, \cdots, \nu$, $\beta_j(\epsilon)$, $j = r + 1, \cdots, n$, have finite derivatives at $\epsilon = 0$, i.e., $\alpha_j'(0)$, $\sigma_j'(0)$, $j = 1, \cdots, \nu$, $\beta_j'(0)$, $j = r + 1, \cdots, n$, exist.

The determining system (7) is now made up of $2\nu + (n - \mu)$ equations, and in these equations certainly the sets of integers $a = (a_1, \cdots, a_\nu)$, $b = (b_1, \cdots, b_\nu)$, and the $2\nu + (n - \mu)$ complex indeterminates c_j should appear. It is actually convenient to replace the latter by real indeterminates $\lambda = (\lambda_1, \cdots, \lambda_\nu)$, amplitudes $\theta = (\theta_1, \cdots, \theta_\nu)$, phases, $\eta = (\eta_1, \cdots, \eta_{n-r})$, amplitudes [7, 13]. Then equations (7) assume the form

$$\begin{aligned} \epsilon P_j &= \alpha_j(\epsilon), \quad \epsilon Q_j = a_j b_j^{-1}\omega - \gamma_j(\epsilon), \quad j = 1, \cdots, \nu, \\ \epsilon R_j &= \beta_{r+j}(\epsilon), \quad j = 1, \cdots, n - r, \end{aligned} \tag{17}$$

where $P_j(a, b, \lambda, \theta, \eta, \omega, \epsilon)$, $Q_j(\cdots)$, $R_j(\cdots)$ are continuous functions of the parameters λ, θ, η, ϵ (and ω if (16) is autonomous), depending upon the functions f_j. Under a Lipschitz condition (6) the same P_j, Q_j, R_j are determined by the inherent method of successive approximations. The same functions, for $\epsilon = 0$, are given at the first step of the process, but it is useful to know that they are given also by simple quadratures:

$$\begin{aligned} &P_j(a, b\ \lambda, \theta, \eta, \omega, 0) \\ &\quad = (\lambda_j Tb_0^{-1})\left[\cos\theta_j \int_0^{Tb_0} f_j \cos a_j\, b_j^{-1}\, \omega\, u du - \sin\theta_j \int_0^{Tb_0} f_j \sin a_j b_j^{-1}\, \omega u du\right], \\ &Q_j(a, b, \lambda, \theta, \eta, \omega, 0) \\ &\quad = (\lambda_j Tb_0^{-1})\left[-\sin\theta_j \int_0^{Tb_0} f_j \cos a_j\, b_j^{-1}\, \omega u du - \cos\theta_j \int_0^{Tb_0} f_j \sin a_j b_j^{-1}\, \omega u\right] du, \\ &R_j(\alpha, b, \lambda, \theta, \eta, \omega, 0) = (\eta_j\, Tb_0)^{-1} \int_0^{Tb_0} f_{r+j}\, du, \end{aligned} \tag{18}$$

where the arguments of the f's are $x_j = b_j\tau_j^{-1} \sin (\tau_j t + \theta_j)$, $j = 1, \cdots, \nu$, $x_j = 0, j = \nu + 1, \cdots, \mu$, and $j = \mu + 1, \cdots, r$, $x_j = \eta_{j-r}, j = r + 1, \cdots, n$.

(i) For $|\epsilon|$ sufficiently small, and any solution λ, θ, η of system (17) there is a periodic solution of system (16) of the form

$$
\begin{aligned}
&x_j(t, \epsilon) = \lambda_j a_j^{-1} b_j \omega \sin(a_j b_j^{-1} \omega t + \theta_j) + 0(\epsilon), \quad j = 1, \cdots, \nu, \\
(19) \qquad &x_j(t, \epsilon) = 0(\epsilon), \quad j = \nu + 1, \cdots, r, \\
&x_j(t, \epsilon) = \eta_{j-r} + 0(\epsilon), \quad j = r + 1, \cdots, n.
\end{aligned}
$$

For $|\epsilon|$ small the terms $\lambda_j a^{-1} b_j \omega \sin(a_j b^{-1} \omega t + \theta_j)$ are predominant. If all $a_j = b_j = 1$ we say that (19) is a harmonic solution, otherwise (19) is said to be a subharmonic ($a_j = 1, b_j > 1$), or ultraharmonic solution, etc.

Since $\alpha_j(0) = 0, \gamma_j(0) = \sigma_j(0) = \tau_j = a_j b_j^{-1}\omega, \beta_{r+j}(0) = 0$, system (17), dividing each equation by ϵ and taking the limit as $\epsilon \to 0$, yields:

$$
\begin{aligned}
&P_{j0} = \alpha_j'(0), \quad j = 1, \cdots, \nu, \\
(20) \qquad &Q_{j0} = \gamma_j'(0), \quad j = 1, \cdots, \nu, \\
&R_{j0} = \beta_{r+j}'(0), \quad j = 1, \cdots, n - r,
\end{aligned}
$$

where $P_{j0} = P_j(a, b, \lambda, \theta, \eta, \omega, 0)$, $Q_{j0} = Q_j(\cdots)$, $R_{j0} = R_j(\cdots)$, are given by (18).

By the use of Brouwer's fixed point theorem the following theorem ([7]) is proved:

(ii) If system (20) has a solution $\lambda_0, \theta_0, \eta_0$, if we can find an interval $I = [\lambda_j' \le \lambda_j \le \lambda_j'', \theta_j' \le \theta_j \le \theta_j''\ j = 1, \cdots, \nu, \eta_j' \le \eta_j \le \eta_j'', j = 1, \cdots, n - r$, of center $(\lambda_0, \theta_0, \eta_0)$, and an ordering of the $2\nu + (n - r)$ functions $F_j = P_{j0} - \alpha_j'(0), j = 1, \cdots, \nu, F_{\nu+j} = Q_{j0} - \gamma_j'(0), j = 1, \cdots, \nu, F_{2\nu+j} = R_{j0} - \beta_{r+j}, j = 1, \cdots, n - r$, such that F_s has opposite constant signs on the corresponding opposite two faces of I, $s = 1, \cdots, 2\nu + n - r$, then (17) has a solution $(\lambda, \theta, \eta) \in I$ for every $|\epsilon|$ sufficiently small, and, by force of (i), system (16) has a periodic solution of the form (19) for all $|\epsilon|$ sufficiently small.

Finally, by the use of properties of the topological index, the following statement ([7]) is proved under differentiability conditions of the functions P_{j0}, Q_{j0}, R_{j0} only:

(iii) If system (20) has a solution $\lambda_0, \theta_0, \eta_0$, if the functions P_{j0}, Q_{j0}, R_{j0} have continuous first partial derivatives with respect to $\lambda_1, \cdots, \lambda_\nu, \theta_1, \cdots, \theta_\nu, \eta_1, \cdots, \eta_{n-r}$, in a neighborhood of $(\lambda_0, \theta_0, \eta_0)$ with Jacobian $\neq 0$ at $(\lambda_0, \theta_0, \eta_0)$, then for all $|\epsilon|$ sufficiently small system (17) has at least one solution λ, θ, η, and (16) has at least one periodic solution of the form (19) for all $|\epsilon|$ sufficiently small.

In [13] the following examples, among others, are discussed without difficulties: the nonlinear Mathieu equation with large forcing terms

$$(21) \qquad x'' + \sigma^2 x = A \cos 2\omega t + \epsilon(Ax \cos 2\omega t + Bx^3);$$

the van der Pol equation

$$x'' + \sigma^2 x = \epsilon(1 - x^2)x' + \epsilon p\omega \cos(\omega t + \alpha); \tag{22}$$

the generalized van der Pol equation

$$x'' + \sigma^2 x = \epsilon(1 - x^{2m})x' + \epsilon p\omega \cos(\omega t + \alpha), \tag{23}$$

with m integer and large ("almost square" characteristic function); the system of two nonlinear Mathieu equations

$$\begin{aligned} x'' + \sigma_1^2 x &= \epsilon(Ax + Bx \cos t + Cx^3 + Dxy^2) \\ y'' + \sigma_2^2 y &= \epsilon(Ey + Fy \cos t + Gy^3 + Hx^2 y); \end{aligned} \tag{24}$$

§4. Existence theorems for cycles of autonomous systems

Let us consider now system (16) where the functions f's do not depend on t, i.e. the autonomous real system

$$\begin{aligned} x_j'' + 2\alpha_j x_j' + \sigma_j^2 x_j &= \epsilon f_j(x, x', \epsilon), \quad j = 1, \cdots, \mu, \\ x_j' + \beta_j x_j &= \epsilon f_j(x, x', \epsilon), \quad j = \mu + 1, \cdots, n. \end{aligned} \tag{28}$$

The hypotheses listed at the beginning of §3 reduce now to the following ones: (1) the functions f_j, $j = 1, \cdots, n$, are Lipschitzian in (x, x', ϵ) for $|x_j| \le R$, $j = 1, \cdots, n$, $|x_j'| \le R$, $j = 1, \cdots, \mu$, $|\epsilon| \le \epsilon_0$; (2) For some $\omega_0 > 0$, and integers $0 \le \nu \le \mu \le r \le n$, we have $\alpha_j(0) = 0$, $\sigma_j(0) = a_j\omega_0/b_j$, $a_j, b_j > 0$ integers, $j = 1, \cdots, \nu$, $\rho_{j1}(0), \rho_{j2}(0) \ne im\omega_0/b_0$, $b_0 = b_1 \cdots b_\nu$, $j = \nu + 1, \cdots, \mu$, $m = 0, \pm 1, \cdots, \beta_j(0) \ne 0, j = \mu + 1, \cdots, r, \beta_j(0) = 0, j = r + 1, \cdots, n$. As usual all $\sigma_j > 0$, α_j, β_j are continuous functions of ϵ, or constants, and $\alpha_j'(0)$, $\sigma_j'(0)$, $j = 1, \cdots, \nu$, $\beta_j'(0)$, $j = r + 1, \cdots, n$, exist. In the present situation $\omega = \omega(\epsilon)$ can be thought of as an undetermined continuous function of ϵ to be added to the unknowns in system (17). On the other hand, we must expect that one of the phases, say θ_1, remains arbitrary, since (28) is autonomous. In other words, we may try, for instance, to solve system (17) for the $2\nu + (n - r)$ unknowns $\lambda_1, \cdots, \lambda_\nu, \theta_2, \cdots, \theta_\nu, \eta_1, \cdots, \eta_{n-r}, \omega$, leaving θ_1 arbitrary. But $\lambda = (\lambda_1, \cdots, \lambda_\nu)$, $\theta = (\theta_2, \cdots, \theta_\nu)$, $\eta = (\eta_1, \cdots, \eta_{n-r})$. The corresponding periodic solutions of system (28) (cycles) have still the form (19), but now ω depends on ϵ. Instead of system (20) it is convenient to consider the analogous system obtained by (17) by first dividing all but one equation (17) by ϵ and taking the limit as $\epsilon \to 0$, say the system

$$\begin{aligned} F_j &\equiv P_{j0} - \alpha_j'(0) = 0, \quad j = 1, \cdots, \nu, \\ F_{\nu+1} &\equiv a_1\omega/b_1 - \gamma_1(0) = 0, \\ F_{\nu+j} &\equiv Q_{j0} - \gamma_j'(0), \quad j = 2, \cdots, \nu, \\ F_{2\nu+j} &\equiv R_{j0} - \beta_{r+j}'(0), \quad j = 1, \cdots, n - r. \end{aligned}$$

The statements corresponding to (i), (ii), (iii) read as follows (see [7, 25] for the present hypotheses, and [13, 16] for the analytic case).

(iv) For $|\epsilon|$ sufficiently small, and any solution $\lambda, \theta, \eta, \omega$ of system (17), $\omega(0) = \omega_0$, there is a periodic solution (cycle) of system (28) of the form (19).

(v) If system (29) has a solution $\lambda_0, \theta_0, \eta_0, \omega_0$, if we can find an interval $I = [\lambda_j' \le \lambda_j \le \lambda_j'', j = 1, \cdots, \nu, \theta_j' \le \theta_j \le \theta_j'', j = 2, \cdots, \nu, \eta_j' \le \eta_j \le n_j'', j = 1, \cdots, n - r, \omega' \le \omega \le \omega'']$ of center $(\lambda_0, \theta_0, \eta_0, \omega_0)$, and an ordering of the $2\nu + (n - r)$ functions $F_j = P_{j0} - \alpha_{j0}'(0)$, $j = 1, \cdots, \nu$, $F_{\nu+j} = Q_{j0} - \gamma_j'(0)$, $j = 1, \cdots, \nu$, $F_{2\nu+j} = R_{j0} - \beta_{r+j}$, $j = 1, \cdots, n - r$, such that F_s has opposite constant signs on the corresponding opposite two faces of I, $s = 1, \cdots, 2\nu + n - r$, then (17) has a solution $(\lambda, \theta, \eta, \omega) \in I$ for every $|\epsilon|$ sufficiently small and system (28) has a cycle of the form (19).

(vi) If system (29) has a solution $\lambda_0, \theta_0, \eta_0, \omega_0$, if the functions P_{j0}, Q_{j0}, R_{j0} have continuous partial derivatives with respect to $\lambda_1, \cdots, \lambda_\nu$, $\theta_2, \cdots, \theta_\nu, \eta_1, \cdots, \eta_{n-r}, \omega$ in a neighborhood of $(\lambda_0, \theta_0, \eta_0, \omega_0)$ with Jacobian $\neq 0$ at $(\lambda_0, \theta_0, \eta_0, \omega_0)$, then for all $|\epsilon|$ sufficiently small system (17) has a solution $\lambda, \theta, \eta, \omega$ and (28) has a cycle of the form (19).

For $\nu = 1, r = n$, we may well take $a_1 = b_1 = 1$, and then $\alpha_1(0) = 0$, $\omega_0 = \sigma_1(0) = \gamma_1(0)$, $\rho_{j1}(0), \rho_{j2}(0) \neq im\,\omega_0$, $j = 2, \cdots, \mu$, $m = 0, \pm 1, \cdots$, $\beta_j(0) \neq 0$, $j = \mu + 1, \cdots, n$. In this situation there are only two functions $P = P_1$, $Q = Q_1$, and there is only one amplitude $\lambda = \lambda_1$, and one phase $\theta = \theta_1$, which must remain arbitrary. Finally (29) becomes

$$F_1 \equiv P(\lambda, \omega, \epsilon) = 0, \quad F_2 \equiv \omega - \epsilon Q(\lambda, \omega, \epsilon) - \gamma_1(\epsilon) = 0, \tag{30}$$

with

$$P(\lambda, \omega, 0) = (T\lambda)^{-1} \int_0^T f_1 \cos \omega t \, dt, \; Q(\lambda, \omega, 0) = (T\lambda)^{-1} \int_0^T f_1 \sin \omega t \, dt, \tag{31}$$

with $T = 2\pi/\omega$, and where the arguments of f_1 are $\lambda\omega^{-1} \sin \omega t, 0, \cdots, 0, \lambda \cos \omega t, 0, \cdots, 0, 0$. For $\epsilon = 0$ the second equation is identically satisfied by $\omega = \omega_0$, and if ω', ω'' are any two numbers $\omega' < \omega_0 < \omega''$ we certainly have $F_2 < 0$ for $\omega = \omega'$, $F_2 > 0$ for $\omega = \omega''$, $\epsilon = 0$. Then (as a corollary of the 2-dim. Brouwer fixed point theorem) we have, instead of (v),

(vii) If $\nu = 1, r = n$, and we can find a pair $\lambda' < \lambda''$ such that $P(\lambda', \omega_0, 0) < 0$, $P(\lambda'', \omega_0, 0) > 0$, then there is a solution (λ, ω) of (17), $\lambda' < \lambda < \lambda''$, $\omega' < \omega < \omega''$, for all $|\epsilon|$ sufficiently small and system (28) has a cycle of the form

$$x_1 = \lambda\omega \sin(\omega t + \theta) + 0(\epsilon), x_j = 0(\epsilon), \quad j = 2, \cdots, n, \quad \theta \text{ arbitrary}, \quad \omega = \omega_0 + 0(\epsilon). \tag{32}$$

Note that $F_2(\lambda, \omega, 0) \equiv \omega$, $\partial F_2/\partial\omega \equiv 1$, $\partial F_2/\partial\lambda \equiv 0$. Hence (vi) reduces to

(viii) If $\nu = 1, r = n$, if λ_0 is a root of the equation $P(\lambda, \omega_0, 0) = 0$, if $P(\lambda, \omega, 0)$ has continuous first partial derivatives in a neighborhood of (λ_0, ω_0) and $\partial P/\partial\lambda \neq 0$ there, then the Jacobian of system (30) is $= \partial P/\partial\lambda \neq 0$, system (17) has a solution (λ, ω) for all $|\epsilon|$ sufficiently small, and (28) has a cycle as in (vii).

Furthermore, the determination of $P(\lambda, \omega, 0)$ by means of (31) may be simplified by the following remarks. If we put $Z(z_1, x_1') = f_1(x_1, 0, \cdots, 0, x_1', 0, \cdots, 0, 0)$, $g(x, x', \epsilon) = f(x, x', \epsilon) - Z(x_1, x_1')$ and we decompose

$$Z(x_1, x_1') = Z_1(x_1, x_1') + Z_2(x_1, x_1') + Z_3(x_1, x_1') + Z_4(x_1, x_1')$$

into its odd and even parts, say Z_1 even in x_1 and odd in x_1', etc., we have $g(x_1, 0, \cdots, 0, x_1', 0, \cdots, 0, 0) = 0$, $f(x, x', \epsilon) = g(x, x', \epsilon) + Z_1(x_1, x_1') + Z_2(x_1, x_1') + Z_3(x_1, x_1') + Z_4(x_1, x_1')$, and only $Z_1(x_1, x_1')$ has any bearing on $P(\lambda, \omega, 0)$, i.e.

$$(33) \qquad P(\lambda, \omega, 0) = (T\lambda)^{-1} \int_0^T Z_{11}(\lambda\omega^{-1} \sin \omega t, \quad \lambda \cos \omega t) \cos \omega t \, dt.$$

In case Z_{11} is a polynomial in x_1, x_1', i.e., $Z_{11} = \sum a_{hk} x_1^{2h} x_1'^{2k-1}$, $h \geq 0$, $k \geq 1$, then

$$(34) \qquad P(\lambda, \omega, 0) = 2^{-2} \sum a_{hk} \frac{(2h)!\,(2k)!}{h!\,k!\,(h+k)!} \left(\frac{\lambda}{2\sigma_1}\right)^{2h+2k-2}$$

with $\omega_0 = \sigma_1 = \sigma_1(0)$. If, for instance a_{01} and the coefficient of maximal power of λ in Z_{11} are ($\neq 0$) and of opposite signs, then certainly there is at least one root λ of odd order for the equation $P(\lambda, \omega_0) = 0$.

As an example one may well consider the system ([7], [19])

$$(35) \qquad \begin{aligned} x_1'' + \sigma_1^2 x_1 &= \epsilon[Z_1(x_1, x_1') + Z_2(x_1, x_1') + Z_3(x_1, x_1') + Z_4(x_1, x_1') + g(x, x', \epsilon)], \\ x_j'' + 2\alpha_j x_j' + \sigma_j^2 x_j &= \epsilon f_j(x, x', \epsilon), \quad j = 2, \cdots, \mu, \\ x_j' + \beta_j x_j &= \epsilon f_j(x, x', \epsilon), \quad j = \mu + 1, \cdots, n, \end{aligned}$$

with Z_1, Z_2, Z_3, Z_4, g as above, with $\sigma_1(0) = \omega_0 > 0$, $\sigma_j(0) > 0$, $j = 2, \cdots, \nu$, $\beta_j(0) \neq 0$, $j = \nu + 1, \cdots, n$, and either $\alpha_j(0) \neq 0$, or $\alpha_j(0) = 0$, $\sigma_j(0) \neq m\omega_0$, $m = 0, 1, 2, \cdots$, $j = 2, \cdots, \mu$. If we take $Z_{11} = (1 - x_1^2)x_1'$ then, by (34), $P = (\frac{1}{2})(1 - \lambda^2/4\sigma_1^2)$ and (35) has a cycle of the form

$$(36) \quad x_1 = \lambda\omega^{-1} \sin(\omega t + \theta) + 0(\epsilon), \quad x_j = 0(\epsilon). \quad j = 2, \cdots, n,$$

with $\lambda = 2\sigma_1(0) + 0(\epsilon)$, $\omega = \sigma_1(0) + 0(\epsilon)$. For $Z_{12} = Z_{13} = Z_{14} = g = 0$, $\mu = n = 1$, (35) reduces to the autonomous van der Pol equation $x_1'' + x_1 = \epsilon(1 - x_1^2)x_1'$. If we take $Z_{11} = (1 - x_1^2 - x_1'^2)x_1'$, then $P = (\frac{1}{2})(1 - \lambda^2/\sigma_1^2)$ and (35) has a cycle of the form (36) with $\lambda = \sigma_1(0) + 0(\epsilon)$, $\omega = \sigma_1(0) + 0(\epsilon)$. If we take $Z_{11} = (1 - |x_1|)x_1'$, then by using (33) we have $P = (T\lambda)^{-1} \int_0^T (1 - \lambda\omega^{-1} |\sin \omega t|) \cos^2 \omega t \, dt = (\frac{1}{2})(1 - \lambda/\pi\sigma_1)$ and (35) has a cycle (36) with $\lambda = \pi\sigma_1(0) + 0(\epsilon)$, $\omega = \sigma_1(0) + 0(\epsilon)$. Note that no hypotheses are made on the functions $f_2, \cdots, f_n, Z_2, Z_3, Z_4, g$, but the standard ones above and continuity as in §1.

As another example we may consider the system

$$(37) \qquad \begin{aligned} x'' + x - \epsilon(1 - x^2 - y^2)x' &= \epsilon f_1(x, y, y') + \epsilon g_1(x, x', y, y')y, \\ y'' + 2y - \epsilon(1 - x^2 - y^2)y' &= \epsilon f_2(x, x', y) + \epsilon g_2(x, x', y, y')x, \end{aligned}$$

where $f_1(-x, y, y') = -f_1(x, y, y')$, $f_2(x, x', -y) = f_2(x, x', y)$. By putting either $x = x_1$, $y = x_2$, or $y = x_1$, $x = x_2$, and applying the considerations above we find that (37) has two cycles ([16]) given by

$$x = \lambda \sin (\omega t + \theta) + 0(\epsilon), \quad y = 0(\epsilon), \quad \lambda = 2 + 0(\epsilon), \quad \omega = 1 + 0(\epsilon),$$

$$x = 0(\epsilon), \quad y = \lambda \sin (\omega t + \theta) + 0(\epsilon), \quad \lambda = 2^{\frac{3}{2}} + 0(\epsilon), \quad \omega = 2^{\frac{1}{2}} + 0(\epsilon).$$

Both cycles are asymptotically orbitally stable.

All theorems of the present section have been extended in [25] to the case where the second members of equations (28) are finite sums of functions of the type

$$\sum_{s=1}^{k} f_{js}[x(t - \lambda_s), x'(t - \lambda_s), \epsilon]$$

where $\lambda_1, \cdots, \lambda_k \geq 0$ are arbitrary numbers (lags). For instance the van der Pol equation with a small parameter ϵ and a lag λ,

$$x''(t) + x(t) = \epsilon[1 - x^2(t)]x'(t) + \epsilon p[1 - x^2(t - \lambda)]x'(t - \lambda),$$

has a periodic solution similar to the one of the usual van der Pol equation $(p = 0)$.

§5. Systems presenting symmetries

We shall now consider systems (16) with $0 \leq \nu \leq \mu = r \leq n$, $\alpha_j \equiv 0$, $j = 1, \cdots, \mu$, $\beta_j \equiv 0$, $j = \mu + 1, \cdots, n$. In other words, we consider systems

$$(38) \qquad \begin{aligned} x_j'' + \sigma_j^2 x_j &= \epsilon f_j(x, x', t, \epsilon), \quad j = 1, \cdots, \mu, \\ x_j' &= \epsilon f_j(x, x', t, \epsilon), \quad j = \mu + 1, \cdots, n, \end{aligned}$$

with $\sigma_j(0) = a_j\omega/b_j$, $\omega > 0$, a_j, $b_j > 0$ integers, $j = 1, \cdots, \nu$, $0 < \sigma_j(0) \neq m\omega/b_0$, $b_0 = b_1 b_2 \cdots b_\nu$, $m = 0, 1, 2, \cdots$, $j = \nu + 1, \cdots, \mu$. Then we have $2\nu + (m - \mu)$ functions P_j, Q_j, $j = 1, \cdots, \nu$, $R_{\mu+j}$, $j = 1, \cdots, n - \mu$, and parameters $\lambda = (\lambda_1, \cdots, \lambda_\nu)$, $\theta = (\theta_1, \cdots, \theta_\nu)$, $\eta = (\eta_1, \cdots, \eta_{n-\mu})$. We suppose that, for some m, $0 \leq m \leq \nu$, and all $u = (x_1, \cdots, x_m)$, $v = (x_{m+1}, \cdots, x_\mu)$, $w = (x_{\mu+1}, \cdots, x_n)$, we have

$$(39) \qquad \begin{aligned} f_j(u, -v, w, -u', v', -t) &= f_j(u, v, w, u', v', t), \quad j = 1, \cdots, m, \\ f_j(u, -v, w, -u', v', -t) &= -f_j(u, v, w, u', v', t), \quad j = m + 1, \cdots, n. \end{aligned}$$

In other words, all f_j, $j = 1, \cdots, m$, are even in the vector (v, u', t), and all others f_j, $j = m + 1, \cdots, n$, are odd in the same vector. Thus, for $m = 0$, all f_j are odd in the same vector. Under these hypotheses it was shown [7, 19] that, by taking $\theta_j = \pi/2$, $j = 1, \cdots, m$, $\theta_j = 0$, $j = m + 1, \cdots, \nu$, all P_j, R_j are identically zero. Thus the determining system (17) reduces to only ν equations

$$(40) \qquad a_j b_j^{-1} \omega - \epsilon Q_j(a, b, \lambda, \eta, \omega, \epsilon) - \sigma_j(\epsilon) = 0, \quad j = 1, \cdots, \nu,$$

in the $\nu + (n - \mu)$ unknown amplitudes $\lambda = (\lambda_1, \cdots, \lambda_\nu)$, $\eta = (\eta_1, \cdots,$

$\eta_{n-\mu}$). Thus system (40) may present some degree of indetermination, and, in correspondence, (38) will have a family of periodic solutions of the form (deduced from (19)):

$$(40)\qquad \begin{aligned} x_j &= \lambda_j a_j^{-1} b_j \omega \cos a_j b_j^{-1} \omega t + 0(\epsilon), & j &= 1, \cdots, m, \\ x_j &= \lambda_j a_j^{-1} b_j \omega \sin a_j b_j^{-1} \omega t + 0(\epsilon), & j &= m+1, \cdots, \nu, \\ x_j &= 0(\epsilon), & j &= \nu+1, \cdots, \mu, \\ x_j &= \eta_{j-\mu} + 0(\epsilon), & j &= \mu+1, \cdots, n. \end{aligned}$$

By division by ϵ and limit as $\epsilon \to 0$ equations (40) become

$$(41)\qquad Q_{j0} - \sigma_j'(0) = 0, \quad j = 1, \cdots, \nu.$$

For Q_{j0} we have here, instead of (18), the expressions

$$Q_{j0} = (T\lambda_j)^{-1} \int_0^T f_j(\cdots) \cos a_j\, b_j^{-1}\, \omega t\, dt, \quad j = 1, \cdots, m,$$

$$Q_{j0} = (T\lambda_j)^{-1} \int_0^T f_j(\cdots) \sin a_j\, b_j^{-1}\, \omega t\, dt, \quad j = m+1, \cdots, \nu,$$

where the arguments of f_j are the x_j' s given in (40) for $\epsilon = 0, j = 1, \cdots, n$, the corresponding derivatives x_j', $j = 1, \cdots, \nu$, and t and $\epsilon = 0$. We give here only a statement corresponding to (iii) (see [19]):

(ix) If system (41) has a solution $\lambda_0(\eta)$ for every η in a compact set G, if for every η in G the functions Q_{j0} have continuous first partial derivatives with respect to $\lambda_1, \cdots, \lambda_\nu$ with Jacobian $\neq 0$, then for all $|\epsilon|$ sufficiently small and all η in G, equations (40) have a solution $\lambda(\eta, \epsilon)$, and then system (38) has a family of periodic solutions (40) with $\lambda = \lambda(\eta, \epsilon)$ depending upon η in G, for all $|\epsilon|$ sufficiently small.

Obviously under these conditions the Jacobian of order $2\nu + (n - \mu)$ considered in §3 is identically zero. A good example for (ix) is the third order equation ([19])

$$(42)\qquad y''' + \sigma^2 y' = \epsilon f(y, y', y'', t, \epsilon)$$

with f periodic in t of period $T = 2\pi/\omega$, $\omega > 0$, with $f(y, -y', y'', -t, \epsilon) = -f(y, y', y'', t, \epsilon)$, $\sigma(0) = a\omega/b$, $a, b > 0$ integers. Then (42) has a one parameter family of periodic solutions of the form $y = c_1 a\omega b^{-1}\sigma^{-2} \cos a\, b^{-1}\omega t + c_2 + 0(\epsilon)$, $c_1 = c_1(c_2, \epsilon)$, for all $|\epsilon|$ sufficiently small provided there is a solution $c_{10} = c_{10}(c_2)$ of the equation $H = 0$ with $\partial H/\partial c_1 \neq 0$ and

$$H = (c_1 T)^{-1} \int_0^T f[y(t, 0), y'(t, 0), y''(t, 0), t, 0] \sin ab^{-1}\, \omega t\, dt.$$

The autonomous case leads to analogous results. Nevertheless we mention here only the following one which is an explicit existence theorem of families of periodic solutions ([7], [19]).

(x) Consider an autonomous system

$$(43)\qquad \begin{aligned} x_j'' + \sigma_j^2 x_j &= \epsilon f_j(x, x', \epsilon), \quad j = 1, \cdots, \mu, \\ x_j' &= \epsilon f_j(x, x', \epsilon), \quad j = \mu + 1, \cdots, n, \end{aligned}$$

where the functions f_j, $j = 1, \cdots, n$, are Lipschitzian in the $n + \mu$ variables $x_1, \cdots, x_n, x_1', \cdots, x_\mu'$, and continuous in ϵ, for $|x_s| \leq R$, $s = 1, \cdots, n$, $|x_s'| \leq R$, $s = 1, \cdots, \mu$, and $|\epsilon| \leq \epsilon_0$. In addition suppose for f_1 only, that $f_1(0, x_2, \cdots, x_n, 0, x_2', \cdots, x_\mu', \epsilon) = 0$. Suppose that either all f_j, $j = 1, \cdots, n$, are odd in $(x_1, \cdots, x_\mu)$, or f_1 is even and $f_2, \cdots, f_n$ are odd in $(x_2, \cdots, x_\mu, x_1')$. Suppose $\sigma_j(\epsilon) > 0$, $j = 1, \cdots, \mu$, are continuous functions of ϵ [or constants] and $\sigma_j(0) \neq m\,\sigma_1(0)$, $m = 0, 1, 2, \cdots, j = 2, \cdots, \mu$. Put $\omega_0 = \sigma_1(0)$ and let $r_2 < R$. Then there exists an ϵ_1, $0 < \epsilon_1 \leq \epsilon_0$, such that for all real ϵ, $\lambda_1, \eta_1, \cdots, \eta_{n-\mu}$, $|\epsilon| \leq \epsilon_1$, $|\lambda_1|, |\eta_1|, \cdots, |\eta_{n-\mu}| \leq r_2$, system (43) has a (real) cycle of the form

$$(44)\qquad \begin{aligned} &x_1(t, \epsilon) = \lambda_1\omega^{-1}\cos\omega t + 0(\epsilon), \quad \text{or} \quad x_1(t, \epsilon) = \lambda_1\omega^{-1}\sin\omega t + 0(\epsilon), \\ &x_j(t, \epsilon) = 0(\epsilon), \quad j = 2, \cdots, \mu, \\ &x_j(t, \epsilon) = \eta_{j-\mu} + 0(\epsilon), \quad j = \mu + 1, \cdots, n, \end{aligned}$$

where x_1 is even (or odd) in t, $x_2, \cdots, x_\mu$ are odd, $x_{\mu+1}, \cdots, x_n$ and even, where $\omega = \omega(\epsilon, \lambda_1, \eta_1, \cdots, \eta_{n-\mu})$ is a continuous function of these parameters, and $\omega = \omega_0$ for $\epsilon = 0$. Also t can be replaced by $t + \theta$, θ arbitrary. Thus (43) has a $(n - \mu + 2)$-parameter family of cycles.

As a first example we may consider the simple equation $x'' + x = \epsilon f(x, x')$ with $f(0, 0) = 0$ and either $f(x, -x') = f(x, x')$, or $f(-x, x') = -f(x, x')$. This equation has a family of cycles of the form $x = \lambda\omega^{-1}\cos(\omega t + \theta) + 0(\epsilon)$, or $x = \lambda\omega^{-1}\sin(\omega t + \theta) + 0(\epsilon)$, with $\omega = \omega(\lambda, \epsilon) = 1 + 0(\epsilon)$, λ, θ arbitrary, $|\epsilon|$ sufficiently small. We may take for instance, $f = x + x^2 + x'^2$, or $f = |x| + |x'|$, or $f = |x|x'$.

As another example we may consider the system

$$x'' + x = \epsilon(1 - |y|)x', \quad y'' + 2y = \epsilon(1 - |x|)y'$$

which has two families of cycles respectively of the forms

$$x = \lambda\omega^{-1}\cos(\omega t + \theta) = 0(\epsilon), \quad y = 0(\epsilon), \quad \omega = \omega(\lambda, \epsilon) = 1 + 0(\epsilon)$$

$$x = 0(\epsilon), \quad y = \lambda\omega^{-1}\cos(\omega t + \theta) + 0(\epsilon), \quad \omega = \omega(\lambda, \epsilon) = 2^{\frac{1}{2}} + 0(\epsilon)$$

λ, θ arbitrary, $|\epsilon|$ sufficiently small.

As a further example let us consider the third order equation ([19])

$$(45)\qquad x''' + \sigma^2 x' = \epsilon f(x, x', x'', \epsilon)$$

where $f(x, -x', x'', \epsilon) = -f(x, x', x'', \epsilon)$. It is proved [19] that (45) has a family of cycles of the form

$$x = -c_1\sigma^{-1} \cos \sigma t + c_2 + 0(\epsilon),$$

with $x(c_1, c_2, \epsilon, -t) = x(c_1, c_2, \epsilon, t)$, where σt can be replaced by $\sigma t + \theta$, c_1, c_2, θ are arbitrary, $| c_1 |, | c_2 | \leq r_2 < R$, and $| \epsilon |$ is sufficiently small. Thus (45) has a 3-parameter family of cycles.

§6. The linear case. Theorems of boundedness

Let us consider linear differential systems of the form

$$(46) \qquad x' = Ax + \epsilon B(t)x, \quad x = (x_1, \cdots, x_n),$$

where ϵ is a parameter, $| \epsilon |$ small, A an $n \times n$-constant matrix, $B(t)$ an $n \times n$ matrix whose elements are periodic in t of period $2\pi/\omega$, L-integrable in $[0, T]$. By Floquet's theory system (46) has a number of solutions of the form $x = e^{\tau t}p(t)$, for convenient numbers τ (characteristic exponents) and periodic vector functions $p(t)$ of period T. A substitution $x = e^{\tau t}u$ with τ indeterminate maps (46) into the system

$$(47) \qquad u' = (A - \tau I)u + \epsilon B(t)u, \quad u = (u_1, \cdots, u_n),$$

I the unit matrix, and thus the particular values of τ for which (47) has a periodic solution $u = p(t)$ of period T are the characteristic exponents. If we apply the approach of §§1, 2, 3 to the determination of these periodic solutions $u = p(t)$ of (47) (and corresponding values τ), we have a natural process for the determination of the solutions of the form $e^{\tau t}p(t)$ of (46) for $| \epsilon |$ sufficiently small.

Note that, if $Y(t, \epsilon)$ is a fundamental system of solutions of (46), then the roots ρ_j, $j = 1, \cdots, n$, of the equation $\det[Y(T, \epsilon) - \rho I] = 0$ are the characteristic multipliers of (46) and the characteristic exponents τ_j, $j = 1, \cdots, n$, are then determined up to multiples of ωi by the relations $\rho_j = e^{\tau_j T}, j = 1, \cdots, n$. The n numbers $\rho_j(0)$ coincide with the characteristic roots of the matrix e^{At} and the functions $\rho_i(\epsilon)$ are continuous in ϵ, even analytic with at most branch points algebraic in character [2]. If some $\rho_s(0)$ is a characteristic root of A of multiplicity k, then there is a group of exactly k multipliers $\rho_j(\epsilon)$ approaching $\rho_s(0)$ as $\epsilon \to 0$. The approach of §§1, 2, 3 applied to (47) makes it possible to determine these groups of multipliers, namely, matrices of order k can be given explicitly whose characteristic roots are the k multipliers $\rho_j(\epsilon)$ ($| \epsilon |$ sufficiently small) [21, 23]. The present approach has led to the following theorem of boundedness under conditions of symmetry, which is similar in form and proof to (ix) and (x) (see [18]):

(xi) Consider the linear real system

$$(49) \qquad \begin{aligned} x_j'' + \sigma_j^2 x_j &= \epsilon f_j(x, x', t, \epsilon), \quad j = 1, \cdots, \mu, \\ x_j' &= \epsilon f(x, x', t, \epsilon), \quad j = \mu + 1, \cdots, n, \end{aligned}$$

where all f_j, $j = 1, \cdots, n$, are *linear* functions of $x_1, \cdots, x_n, x_1', \cdots, x_\mu'$, with coefficients which are functions of t, periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$ (and in absolute value $\leq K(t)$ for some fixed L-integrable function $K(t)$). Suppose $\sigma > 0$, $\sigma_j \pm \sigma_h \neq m\omega$, $j \neq h, j, h = 1, \cdots, \mu$, $m = 0, \pm 1, \cdots$. Suppose that for some $0 \leq m \leq \mu \leq n$, and all $u = (x_1, \cdots, x_m)$, $v = (x_{m+1}, \cdots, x_\mu)$, $w = (x_{\mu+1}, \cdots, x_n)$, relations (39) hold. Then for all $|\epsilon|$ sufficiently small all solutions of (49) are bounded in $(-\infty, +\infty)$.

For $n > \mu$ no other proof is known of this statement besides the one given in 18 by the present approach. However, the particular cases $n = \mu$ and $n = \mu + 1$, could be proved independently. A very elementary proof for $n = \mu$ and $n = \mu + 1$ was given in [20]. (See also [4], §4.5.) For $n = \mu$ another proof has been given by V. A. Yacubovich by the use of properties of the matrix $Y(T, \epsilon)$ already observed by A. Lyapunov. (See [23] for this type of elementary proof.) For $n = \mu$ system (49) can be written in the form

$$x'' + Ax = \epsilon B(t)x + \epsilon C(t)x', \quad x = (x_1, \cdots, x_n), \tag{50}$$

$A = \text{diag}\,(\sigma_1^2, \cdots, \sigma_n^2)$, and conditions ($xi$) mean:

$$B(-t) = B(t), \quad C(-t) = -C(t). \tag{51}$$

Another theorem of the same nature was proved since 1940 in [6] by the same method, namely supposing, instead of (51), that

$$B_{-1}(t) = B(t), \quad C(t) \equiv 0. \tag{51*}$$

Under these conditions system (50) is "canonic", and an improved form of the same statement has been recently reproved in the frame of the theory of canonic systems by J. Moser, and by properties of the matrix $Y(T, \epsilon)$ by V. A. Yacubovich.

The results above have the following interpretation. Let us consider an auxiliary $\omega\epsilon$-plane. For every pair (ω, ϵ), i.e., point of the $\omega\epsilon$-plane, we may ask whether system (50) has solutions $x(t)$ all bounded in $(-\infty, +\infty)$, or not. Accordingly, as the former or the latter occurs, we may divide the $\omega\epsilon$-plane into zones, which usually are called of stability or instability respectively. Note that for $\omega > 0$, $\epsilon = 0$ the solutions of (50) are all bounded in $(-\infty, +\infty)$ but it may occur that points $(\omega, 0)$ are on the boundary of zones of instability, and then the corresponding frequencies ω are called critical. The results above assure that for system (50) with B even and C odd, or B symmetric and $C \equiv 0$, at most the frequencies $\omega = 2\sigma_j/m$, or $\omega = (\sigma_j \pm \sigma_h)/m$, $m = 1, 2, \cdots$, may be critical. For systems (50) with $n = 1$ the same holds under no condition on B or C ([8]).

For instance, the system

$$x'' + \sigma^2 x = \epsilon y \cos t + \epsilon z \cos 2t$$

$$y' = \epsilon x \sin t + \epsilon z \sin 2t$$

$$z' = \epsilon x \sin 2t - \epsilon y \sin t$$

with $\sigma \neq m$, $m = 1, 2, \cdots$, has solutions all bounded in $(-\infty, +\infty)$ for all $|\epsilon|$ sufficiently small because of (xi) with $n = 3$, $\mu = 1$, $m = 1$. The system

$$x'' + \sigma_1^2 x = \epsilon x \cos t + \epsilon y \cos 2t + \epsilon y' \sin t$$

$$y + \sigma_2^2 y = \epsilon x \cos 3t + \epsilon y \cos t + \epsilon x' \sin 2t$$

with $\sigma_1, \sigma_2 > 0$, $2\sigma_1, 2\sigma_2, |\sigma_1 + \sigma_2| \neq m$, $m = 1, 2, \cdots$, has solutions all bounded in $(-\infty, +\infty)$ for $|\epsilon|$ sufficiently small because of (xi) with $n = 2$, $\mu = m = 2$. The same holds for the system

$$x'' + \sigma_1^2 x = \epsilon x \sin 2t + \epsilon y \cos t$$

$$y'' + \sigma_2^2 y = \epsilon x \cos t - \epsilon y \cos t$$

with σ_1, σ_2 as above, because of (51*). The same holds for the second order equation

$$x'' + \sigma^2 x = \epsilon x \sin t + \epsilon x' (\sin t + \cos t),$$

with $2\sigma \neq m$, $m = 1, 2, \cdots$, because of [8].

The independent research mentioned above (J. Moser, V. A. Yacubovich, and also M. G. Krein, I. M. Gelfand and V. B. Lidsky) have reduced the set of possible critical points. For instance the points $(\sigma_j - \sigma_h)/m$ are not critical under conditions (51) or (51*).

The study of single frequencies ω for a given system to establish whether they are critical or not has been initiated in [21] by the present approach and by the authors mentioned above by canonical systems or Lyapunov's arguments (see [23] for a summary of results). Other results by the present approach have been anouncned ([24]).

For systems (50) when $n > 1$ and no symmetry prevails the situation is quite different as it was proved since 1940 in [6] by the present approach as well as later in [11]. Then every frequency is likely to be critical. Namely, in [11] sufficient conditions for this occurrence are proved by the present approach for given systems (50). In other words, if the numbers σ_j are known, as well the matrices $B(t)$, $C(t)$, i.e., the coefficients of the Fourier series of their elements, and ϵ, ω are considered as parameters, then a frequency ω is not critical, or critical, according as a certain infinite set of expressions $M, N, \cdots$, are all zero, or at least one is $\neq 0$ [11]. For instance, for the system

$$x_1'' + \sigma_1^2 x_1 = \epsilon x_2 \sin \omega t$$

$$x_2'' + \sigma_2^2 x_2 = \epsilon x_1 \cos \omega t$$

every frequency ω is critical in the sense that every point $(\omega, 0)$ is point of accumulation of points (ω', ϵ) for which the same system has unbounded solutions in $(0, +\infty)$. The general result just mentioned is similar to a weaker one of V. A. Yacubovich who proved that, for all given $\sigma_1, \cdots, \sigma_n, \omega$ it is possible to choose matrices $B(t)$, $C(t)$ in such a way that the solutions of (50) are not all bounded in $(-\infty, \infty)$ for $|\epsilon| \neq 0$ sufficiently small.

§7. Stability of periodic solutions of nonlinear systems

Given a periodic solution $x = X(t)$ of period $T = 2\pi/\omega$ of a periodic or autonomous system $x' = f(x, t)$, $x = (x_1, \cdots, x_n)$, then the stability of $X(t)$ is related to the characteristic exponents of the linear variational system with periodic coefficients

$$y' = H(t)y, \quad H(t) = \partial f/\partial x\,|_{x=X(t)}\,. \tag{52}$$

If $f(x, t)$ is actually periodic in t of period T (not constant) and all characteristic exponents of (52) have negative real parts, then $x = X(t)$ is asymptotically stable. If f is independent of t, then one of the characteristic exponents is zero ($\equiv 0 \bmod \omega i$), and, if all remaining $n - 1$ ones have real parts negative then $x = X(t)$ is asymptotically orbitally stable (A. Lyapunov). If $f(x, t) = Ax + \epsilon g(x, t, \epsilon)$, where ϵ is a small parameter, then the same parameter appears in (52) and the results mentioned in §6 and others obtained by the same present approach can be applied.

In [3] periodic systems (16) are considered with $\nu = \mu = n$. This is actually the most difficult case (for $n > 1$) since then the characteristic exponents form a unique group of n elements as explained in the first lines of §6. In [3], by using the present approach, explicit expressions are given of the elements of the matrix of order $2n$ whose characteristic roots are the characteristic exponents for $|\,\epsilon\,|$ small. The results have been applied, for instance, to the system of two van der Pol-type equations

$$x'' + x = \epsilon(A - By^2)x' + p\cos t$$
$$y'' + y = \epsilon(C - Dx^2)y' + q\cos 2t$$

(i.e., system (23) with $\sigma_1 = \sigma_2 = 1$, and $a_1 = b_1 = a_2 = b_2 = 1$).

In general the grouping of the characteristic multipliers (§6) is not so bad, and in this most usual situation explicit results have been obtained straightforwardly by the present approach in [22]. We mention here only the following two statements, of which in [23] an idea of the proof is given.

(xii) Let us consider a periodic system (16) with $\mu = n$, $\alpha_j \equiv 0$, $j = 1, \cdots, \mu$, $\nu = 1$, functions f_j of class C', and corresponding periodic solution $x = x_0(t, \epsilon)$ given by (19), $|\,\epsilon\,|$ sufficiently small. Then $\sigma_1 = a_1 b_1^{-1}\omega$, $T_0 = 2\pi b_1/(a_1\omega))$. Suppose $2\sigma_j \neq m\omega$, $|\,\sigma_j \pm \sigma_h\,| \neq m\omega$, $j \neq h$, $j, h = 2, \cdots, \nu$, $m = 1, 2, \cdots$. Then (19) is asymptotically stable as $t \to +\infty$ provided $A_j < 0$, $B > 0$, $j = 1, 2, \cdots$, and $A_1 \neq B$, $A_1^2 \neq B$, where

$$A_j = \int_0^{T_0} f_{jx_j}\,[x_0(t, 0), x_0'(t, 0), t, 0]\,dt, \quad j = 1, 2, \cdots, \nu,$$

$$B = (2T\sigma_1^2)^{-1}\,[\sigma_1^2 A_1^2 + C^2 - D^2 - E^2],$$

$$C = \int_0^{T_0} f_{1x_1}\,dt, \qquad D = \int_0^{T_0} f_{1x_1}\cos 2\sigma_1 t\,dt - \sigma_1 \int_0^{T_0} f_{1x_1'}\sin 2\sigma_1 t\,dt,$$

$$E = \int_0^{T_0} f_{1x_1}\sin 2\sigma_1 t\,dt + \sigma_1 \int_0^{T_0} f_{1x_1'}\cos 2\sigma_1 t\,dt.$$

Except for the condition $A_1 \neq B$, $A_1^2 \neq B$ this generalizes a result of L. Mandelstan and N. Papalexi. This is also a corollary of the result of [3] for $n = 1$.

(xiii) Let us consider an autonomous system (28) with $\nu = 1$, $r = n$, $\alpha_j(0) = 0$, $j = 1, \cdots, \nu$, functions f_j of class C', and corresponding cycle (32), $|\epsilon|$ sufficiently small. (Then $a_1 = b_1 = 1$, $\sigma_1 = \sigma_1(0) = \omega_0$, $T = 2\pi/\omega_0$). Suppose $2\sigma_j \neq m\omega_0$, $|\sigma_j \pm \sigma_h| \neq m\omega_0$, $j \neq h$, $j, h = 2, \cdots, \nu$, $m = 1, 2, \cdots$. Then (32) is asymptotically orbitally stable as $t \to +\infty$ provided

$$A_j = \int_0^T f_{jx'_j}\,[x_0(t,0), x'_0(t,0), 0]\,dt < 0, \quad j = 1, 2, \cdots, \nu.$$

RIAS, Baltimore, Maryland and Purdue University, Lafayette, Indiana

Bibliography

[1] H. R. Bailey. *Harmonics and subharmonics for weakly nonlinear Mathieu type differential equations*. To appear.
[2] H. R. Bailey and L. Cesari, *Boundedness of solutions of linear differential systems with periodic coefficients*. Archive Rat. Mech. and Anal., vol. 1 (1958), 246–271.
[3] H. R. Bailey and R. A. Gambill, *On stability of periodic solutions of weakly nonlinear differential equations*. J. Math. Mech., vol. 6 (1957), 655–668.
[4] L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*. Ergebnisse d. Mathematics und ih. Grenzgebiete, Heft 16 (1959), vii, 271.
[5] L. Cesari, *Un nuovo criterio di stabilità per le soluzioni delle equazioni differenziali lineari*. Annali Scuola Norm. Sup. Pisa (2) 9, (1940), 163–186.
[6] L. Cesari, *Sulla stabilità della soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici*. Mem. Accad. Italia (6) 11, (1941), 633–695.
[7] L. Cesari, *Existence theorems for periodic solutions of nonlinear Lipschitzian differential systems and fixed point theorems*. Contributions to the theory of nonlinear differential equations, 5, 1960, 115-172.
[8] L. Cesari and J. K. Hale, *Second order linear differential systems with periodic L-integrable coefficients*. Riv. Mat. Univ. Parma, vol. 2 (1954), 55–61; vol. 6 (1955), 159.
[9] L. Cesari and J. K. Hale, *A new sufficient condition for periodic solutions of weakly nonlinear differential systems*. Proc. Amer. Math. Soc., vol. 8 (1957), 757–764.
[10] R. A. Gambill, *Stability criteria for linear differential systems with periodic coefficients*. Riv. Mat. Univ. Parma, 5, 1954, 169–181.
[11] R. A. Gambill, *Criteria for parametric instability for linear differential systems with periodic coefficients*. Riv. Mat. Univ. Parma, 6, 1955, 37–43.
[12] R. A. Gambill, *A fundamental system of real solutions for linear differential systems with periodic coefficients*. Riv. Mat. Univ. Parma, 7, 1956, 311–319.
[13] R. A. Gambill and J. K. Hale, *Subharmonic and ultraharmonic solutions for weakly nonlinear systems*. J. Rat. Mech. Anal. 5, 1956, 353–398.
[14] J. K. Hale, *Evaluations concerning products of exponential and periodic functions*. Riv. Mat. Univ. Parma, 5, 1954, 63–81.
[15] J. K. Hale, *On boundedness of the solutions of linear differential systems with periodic coefficients*. Riv. Mat. Univ. Parma, 5, 1954, 137–167.
[16] J. K. Hale, *Periodic solutions of nonlinear systems of differential equations*. Riv. Mat. Univ. Parma, 5, 1954, 281–311.
[17] J. K. Hale, *On a class of linear differential equations with periodic coefficients*. Illinois J. Math., 1, 1957, 98–104.
[18] J. K. Hale, *Linear systems of first and second order differential equations with periodic coefficients*, Illinois J. Math. 2, 1958, 586–591.

[19] J. K. HALE, *Sufficient conditions for the existence of periodic solutions of systems of weakly nonlinear first and second order differential equations*. Journ. Math. Mech., 7, 1958, 163–172.

[20] J. K. HALE. *A short proof of a boundedness theorem for linear differential system with periodic coefficients*. Archive Rat. Mech. Anal., 2, 1959, 429–434.

[21] J. K. HALE, *On the behavior of the solutions of linear periodic differential systems near resonance points*. Contributions to the theory of nonlinear differential equations, 5, 1960, 55–90.

[22] J. K. HALE, *On the stability of periodic solutions of weakly nonlinear periodic and autonomous differential systems*. Contributions to the theory of nonlinear differential equations, 5, 1960, 91–114.

[23] J. K. HALE, *On the characteristic exponents of linear periodic differential systems*. These Proceedings.

[24] C. IMAZ, *Linear differential systems with nonsimple elementary divisors*. Thesis (in preparation) University of Mexico.

[25] W. R. FULLER, *Existence theorems for periodic solutions of systems of differential and differential-difference equations*. Ph.D. Thesis Purdue University, Lafayette, Indiana, 1957.

[26] E. W. THOMPSON. *On stability of periodic solutions of autonomous differential systems*. (To appear.)