PIECEWISE CONTINUOUS DIFFERENTIAL EQUATIONS*

By Johannes André and Peter Seibert

1. Introduction

Certain problems in the theory of automatic control systems gave rise to the investigation of differential equations with piecewise continuous right-hand sides. Special cases have previously been studied from various points of view, e.g., by Bilharz, Bushaw, Flügge-Lotz and the authors[1]. The present paper gives a brief survey of the local properties of the solutions and of the phenomenon called "after-endpoint motion" occurring in systems with retardation. A more detailed exposition of the subject will appear in the "Contributions to the Theory of Nonlinear Oscillations", volume V, edited by S. Lefschetz.

2. The system under consideration

The systems studied in this paper are of the type

$$\text{(S)} \qquad \dot{\mathbf{x}} = d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, \operatorname{sgn} s(\mathbf{x}))$$

where $\mathbf{x} = (x_1, \cdots, x_n)$ is an n-dimensional vector depending on t and sgn $s = s/|s|$ for $s \neq 0$ (sgn 0 is undefined). We assume:

(i) $\mathbf{f}^{\pm}(\mathbf{x}) = \mathbf{f}(\mathbf{x}, \pm 1)$ and $s(\mathbf{x})$ are of the types C^1 and C^2 respectively,

(ii) $s(\mathbf{x})$ and grad $s(\mathbf{x})$ do not vanish simultaneously at any point of the space R^n.

The right-hand side of (S) is discontinuous along the (smooth) hyper-surface

$$\mathbf{S} = \{\mathbf{x} \mid s(\mathbf{x}) = 0\}$$

of R^n, called the *switching space* of (S). Furthermore, we denote

$$\mathbf{D}^{\pm} = \{\mathbf{x} \mid s(\mathbf{x}) \gtrless 0\}.$$

3. Concept of solutions of (S)

In order to define solutions of (S) we consider, together with (S), the pair of systems.

$$(\text{S}^{\pm}) \qquad \dot{\mathbf{x}} = \mathbf{f}^{\pm}(\mathbf{x}).$$

* This research was partially supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract Number AF 49(638)-382. Reproduction in whole or in part is permitted for any purpose of the United States Government.

[1] H. Bilharz, Z. angew. Math. Mech. 22, 206–215 (1942), D. Bushaw, Contributions to the theory of nonlinear oscillations, IV, Princeton 1958; I. Flügge-Lotz, Discontinuous automatic control, Princeton 1953; J. André and P. Seibert, Archiv d. Math. 7, 148–156, 157–165 (1956), Comptes Rendus 245, 625–627 (1957).

DEFINITION.[2] A continuous vector function $\mathbf{x}(t)$ is called a *solution* of (S) if the two following conditions hold:

(I) $\mathbf{x}(t)$ satisfies ($S^{\pm}$) in $\mathbf{D}^{\pm}$ resp.,

(II) $\mathbf{x}(t)$ has only isolated points in $\mathbf{S}$.

(By condition (II), in particular, curves completely contained in $\mathbf{S}$ are excluded as solutions.)

Due to the usual existence and uniqueness theorems for differential equations, there exists exactly one solution through every point outside $\mathbf{S}$. For points of $\mathbf{S}$ (which will be called *switching points*), however, this is not necessarily the case.

4. Types of switching points

A rough classification[3] of the switching points with respect to the local topological behavior of the solutions of (S) yields three principal types.

Denote by $\mathbf{x}^{\pm}(t, \mathbf{u})$ [$\mathbf{u} \in \mathbf{S}$] the solution of ($S^{\pm}$) with the initial point $\mathbf{u}$ (i.e., $\mathbf{x}^{\pm}(0, \mathbf{u}) = \mathbf{u}$). Then the following cases may occur[4]:

$$\text{(a)} \qquad \begin{aligned} \mathbf{x}^{+}(t, \mathbf{u}) &\in \mathbf{D}^{\pm} && \text{for} \quad t \gtrless 0, \\ \mathbf{x}^{-}(t, \mathbf{u}) &\in \mathbf{D}^{\pm} && \text{for} \quad t \gtrless 0, \end{aligned}$$

or vice versa. In this case $\mathbf{u}$ is called a *transition point*. Through every such point there exists exactly one solution of (S), [as in the case of a point outside $\mathbf{S}$].

$$\text{(b)} \qquad \begin{aligned} \mathbf{x}^{+}(t, \mathbf{u}) &\in \mathbf{D}^{\pm} && \text{for} \quad t \gtrless 0, \\ \mathbf{x}^{-}(t, \mathbf{u}) &\in \mathbf{D}^{\pm} && \text{for} \quad t \lessgtr 0. \end{aligned}$$

Here $\mathbf{u}$ is called a *starting point*: Two solutions "start" at $\mathbf{u}$, i.e., they are defined for $t \geq 0$ but not for $t < 0$. No solution starting outside $\mathbf{u}$ can ever reach $\mathbf{u}$.

$$\text{(c)} \qquad \begin{aligned} \mathbf{x}^{+}(t, \mathbf{u}) &\in \mathbf{D}^{\pm} && \text{for} \quad t \lessgtr 0, \\ \mathbf{x}^{-}(t, \mathbf{u}) &\in \mathbf{D}^{\pm} && \text{for} \quad t \gtrless 0. \end{aligned}$$

In this case $\mathbf{u}$ is called an *endpoint*. At every such point two solutions "end", i.e., they are defined for $t \leq 0$ but not for $t > 0$.

All switching points $\mathbf{u}$ at which neither $\mathbf{x}^{+}(t, \mathbf{u})$ nor $\mathbf{x}^{-}(t, \mathbf{u})$ is tangent to $\mathbf{S}$ and which are not critical points for (S^{+}) or (S^{-}), are called *normal*, all others *exceptional*. Clearly, every normal point belongs to one of the classes (a), (b), (c). In general, the exceptional points form a set of measure 0 with respect to $\mathbf{S}$. Among them two cases are of particular interest:

(d) *Bifurcation points*. One of the curves through $\mathbf{u}$, e.g., $\mathbf{x}^{-}(t, \mathbf{u})$, is tan-

[2] An alternative definition was given by Solncev, Moscow. Gos. Univ. Učenye Zapaki, 148, Mat. 4, 144–180 (1951).

[3] The first classification of this kind was given by Solncev, *loc. cit.*

[4] The following formulas are to be understood for sufficiently small $|t|$.

gent to S at $\mathbf{u}$ and contained in $\mathbf{D}^-$ (near $\mathbf{u}$) while the other lies in $\mathbf{D}^+$ for small $t > 0$. Then, apparently, at $\mathbf{u}$ a half trajectory $\{\mathbf{x}^-(t, \mathbf{u}) | t < 0\}$ splits up into the two half trajectories $\{\mathbf{x}^{\pm}(t, \mathbf{u}) | t > 0\}$.

(e) *Fusion points.* Here the situation is the same as in case (d) but with reversed orientation of the trajectories, so that two negative half trajectories have a common continuation.

5. Systems with retardation (switching delay)

In physical systems there is usually a *switching delay* or *retardation*, i.e., the jump from $\mathbf{x}^{\pm}$ to $\mathbf{x}^{\mp}$ takes place a short time after the trajectory passes through S rather than at the precise moment of transition.

To formulate this effect quantitatively we consider a given open set Σ containing S. We split the boundary $\partial\Sigma$ of Σ into the two sets

$$\partial^{\pm}\Sigma = \partial\Sigma \cap \mathbf{D}^{\pm}.$$

On every continuous curve $\mathbf{x}(t)$ defined in a closed interval $I = [0, T]$ or $[0, \infty)$ and satisfying condition (II) of §3, we define a piecewise constant function $e(t)$ uniquely by the conditions:

(A) $e(t) = \pm 1$,

(B) $e(0) = \lim_{t \to +0} \operatorname{sgn} s(\mathbf{x}(t))$

(C) $e(t)$ changes sign at points t' at which $\mathbf{x}(t)$ leaves Σ through $\partial^+\Sigma$ $[\partial^-\Sigma]$, provided that $\lim_{t \to t'-0} e(t) = -1\ [+1]$. Everywhere except at these points it remains constant.

To every system (S) and set Σ we associate the collection of continuous curves $\mathbf{x}(t)$ satisfying condition (II) of §3 and the following:

(I*) For every t at which $e(t)$ is continuous and $= \pm 1$ the function $\mathbf{x}(t)$ satisfies ($S^{\pm}$).

The collection of all these curves $\mathbf{x}(t)$ will be denoted by (S, Σ) and we call every $\mathbf{x}(t)$ a *solution* of (S, Σ). For the system (S, Σ) the following *existence theorem* holds:

Given any point $\mathbf{x}_0 \in R^n$ *there exists a solution of* (S, Σ) *with initial point* $\mathbf{x}_0$ *which is defined for all* $t \geq 0$.

6. After-endpoint motions

The most significant effect of retardation takes place around the domain of endpoints which we denote by $\mathbf{E}(\subseteq \mathbf{S})$. When a trajectory of (S, Σ) reaches an endpoint of (S) the motion, instead of becoming undefined as in the case of the "ideal" system (S), performs oscillations of high frequency (known to engineers under the name of "chattering") around the switching space, the so-called *after-endpoint motion.*

Since every endpoint $\mathbf{u}$ is reached by exactly two motions of (S) (vid. §4 (c)), there exist two after-endpoint motions with $\mathbf{u}$ as initial point which we denote by $\mathbf{x}_-(t, \mathbf{u})$ and $\mathbf{x}_+(t, \mathbf{u})$ respectively (vid. §3).

We now give an approximate representation of the after-endpoint motions for the case that Σ is a narrow zone around **S**. Consider a compact set **U**, contained in the interior of **E**, and assume that for some $\tau > 0$ every trajectory $\mathbf{x}^{\pm}(t, \mathbf{v})$ with $\mathbf{v} \in \mathbf{U}$ leaves Σ at a time $t \in (0, \tau]$.

Then for every $\mathbf{u} \in \mathbf{U}$ *there exists a curve* $\mathbf{x}^*(t, \mathbf{u})$, *contained in* **S**, *such that*

$$\mathbf{x}_{\pm}(t, \mathbf{u}) = \mathbf{x}^*(t, \mathbf{u}) + O(\tau)$$

holds for all t for which $\mathbf{x}^*(t, \mathbf{u}) \in \mathbf{U}$. ($O$ is the usual Landau-symbol applied to every component.) *Moreover,* $\mathbf{x}^*$ *satisfies the following system of differential equations*:

$$(\mathrm{S}^*) \qquad \begin{aligned} \dot{\mathbf{x}} &= P(\mathbf{x})\mathbf{f}^+(\mathbf{x}) \; [=P(\mathbf{x})\mathbf{f}^-(\mathbf{x})] \\ &= \mathbf{f}^+(\mathbf{x}) - \frac{\mathbf{f}^+(\mathbf{x}) \operatorname{grad} s(\mathbf{x})}{(\mathbf{f}^+(\mathbf{x}) - \mathbf{f}^-(\mathbf{x})) \operatorname{grad} s(\mathbf{x})} (\mathbf{f}^+(\mathbf{x}) - \mathbf{f}^-(\mathbf{x})). \end{aligned}$$

The operator $P(\mathbf{x})$ can be interpreted geometrically as a projection of the R^n into the tangential hyperplane of **S** at the point **x** in the direction of

$$\mathbf{f}^+(\mathbf{x}) - \mathbf{f}^-(\mathbf{x}).$$

7. Infinitesimal retardation

Consider a sequence $\{\Sigma_\nu\}$ of sets containing the switching space **S** and a sequence of numbers $\{\tau_\nu\}$ tending to zero such that any solution of $(\mathrm{S}^{\pm})$ starting on **S** will have left Σ_ν by the time τ_ν. Therefore, obviously $\Sigma_\nu \to \mathbf{S}$. Then every sequence $\{\mathbf{x}(t, \mathbf{x}_0 ; \Sigma_\nu)\}$ of solutions of (S, Σ_ν) with a common initial point $\mathbf{x}_0$ tends to a limiting curve $\mathbf{x}^*(t, \mathbf{x}_0)$ for $\nu \to \infty$. The collection of all these limiting curves is called the system with *infinitesimal retardation* (S*) associated to (S). *The restrictions of this system to the sets* $R^n - \mathbf{E}$ *and* **E** *coincide with those of* (S) *and* (S*) *respectively.*

In particular, every curve $\mathbf{x}^* \cap \mathbf{E}$ is the common continuation of uncountably many solutions of (S).

Remark. In the case of a system with constant time-lag τ, defined by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \operatorname{sgn} \mathbf{s}(\mathbf{x}(t - \tau))),$$

the after-endpoint motions tend to the same curves $\mathbf{x}^*$ for $\tau \to 0$ (vid. André-Seibert loc. cit.) as in the case considered here.

Mathematisches Institut, Technische Hochschule, Braunschweig
RIAS, Baltimore, Maryland