SINGULAR PERTURBATION PROBLEMS

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1. Introduction

We are concerned with showing the relationship of the solution of a boundary problem (1.1), (1.2) as $\epsilon \to 0+$ to the solutions of a related degenerate problem (1.3), (1.4). The problems are

$$(1.1)\qquad \left\{\begin{aligned} \frac{d}{dt}\, x_1(t, \epsilon) &= A_{11}(t, \epsilon)x_1 + \cdots + A_{1p}(t, \epsilon)x_p \\ \epsilon^{h_2} \frac{d}{dt}\, x_2(t, \epsilon) &= A_{21}(t, \epsilon)x_1 + \cdots + A_{2p}(t, \epsilon)x_p \\ \vdots \qquad & \qquad \vdots \qquad\qquad\qquad \vdots \\ \epsilon^{h_p} \frac{d}{dt}\, x_p(t, \epsilon) &= A_{p1}(t, \epsilon)x_1 + \cdots + A_{pp}(t, \epsilon)x_p \end{aligned}\right\},$$

$$(1.2)\qquad R(\epsilon)x(a, \epsilon) + S(\epsilon)x(b, \epsilon) = c(\epsilon),$$

$$(1.3)\qquad \left\{\begin{aligned} \frac{dx_1}{dt} &= A_{11}(t, 0)x_1 + \cdots + A_{1p}(t, 0)x_p \\ 0 &= A_{21}(t, 0)x_1 + \cdots + A_{2p}(t, 0)x_p \\ \vdots \quad & \qquad \vdots \qquad\qquad\qquad \vdots \\ 0 &= A_{p1}(t, 0)x_1 + \cdots + A_{pp}(t, 0)x_p \end{aligned}\right\},$$

$$(1.4)\qquad R(0)x(a) + S(0)x(b) = c(0),$$

where the h_i are integers, $0 < h_2 < h_3 < \cdots < h_p = h$, x_i is a vector of dimension n_i, $m = \sum_{j=2}^{p} n_j$, $A_{ij}(t, \epsilon)$ are matrices of appropriate orders with asymptotic expansions, x is the vector $\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$, R and S are square matrices of order $n_1 + m$, and $\epsilon > 0$.

Under three hypotheses, H1, H2, H3, we shall prove Theorem 1 (section 6) which embodies our results for the problem indicated above. We begin by reducing the problem (1.1), (1.2) to a canonical form (2.12), (6.3). We show that the solution of the canonical boundary problem has a limit as $\epsilon \to 0+$ which satisfies the corresponding degenerate differential system and n_1 of the degenerate boundary conditions.

The results of this paper are most closely related to the work of G. G. Chapin, Jr., ([2]), W. R. Wasow ([11]), and I. S. Gradstein ([4]), who consider single Nth order differential equations. Chapin and Wasow consider the case where α conditions are specified at one point and $N - \alpha$ at another point. Gradstein considers an initial value problem which is essentially contained in Theorem 1 of this paper.

2. Preliminary transformations

Let $B_{ii}(t)$ be those matrices which must be non-singular in order to solve the degenerate differential system (1.3) by first solving the last equation for x_p and substituting this solution into the preceding equations and repeating this process until we have only a differential system to solve of the form $dx_1/dt = B_{11}(t)x_1$, $B_{pp}(t) = A_{pp}(t, 0)$, $B_{p-1,p-1}(t) = A_{p-1,p-1} - A_{p-1,p}A_{pp}^{-1}A_{p,p-1}(t, 0)$, etc.

Under the assumption of the non-singularity of these matrices, B_{ii}, $i = 2, \cdots, p$, E. R. Rang, ([8]), has shown there exists a non-singular transformation, $0 \leq \epsilon \leq \epsilon_0$, $a \leq t \leq b$, of the form

$$x(t, \epsilon) = \left(\sum_{i=0}^{L} T_i(t)\epsilon^i\right) y(t, \epsilon), \tag{2.1}$$

which changes the differential system (1.1) into (2.2) with corresponding degenerate form (2.3).

$$\left\{\begin{aligned} \frac{d}{dt}\, y_1(t, \epsilon) &= C_{11}(t, \epsilon)y_1(t, \epsilon) + \cdots + C_{1p}(t, \epsilon)y_p(t, \epsilon) \\ \epsilon^{h_2}\frac{d}{dt}\, y_2(t, \epsilon) &= C_{21}y_1 + \cdots + C_{2p}y_p \\ &\vdots \\ \epsilon^{h_p}\frac{d}{dt}\, y_p(t, \epsilon) &= C_{p1}y_1 + \cdots + C_{pp}y_p \end{aligned}\right\}, \tag{2.2}$$

$$\left\{\begin{aligned} \frac{d}{dt}\, y_1 &= C_{11}(t, 0)y_1 \\ 0 &= C_{22}(t, 0)y_2 \\ &\vdots \\ 0 &= C_{pp}(t, 0)y_p \end{aligned}\right\}, \tag{2.3}$$

where $C_{ii}(t, 0) = B_{ii}(t)$, $B_{ii}(t)$ non-singular $a \leq t \leq b$, and the elements of $C_{ij}(t, \epsilon)$ are $O(\epsilon^\alpha)$ for any particular large integer α, $i \neq j$. We will have occasion to assume that this has been done.

For an Nth order differential system of the form

$$\epsilon^\beta \frac{dz}{dt} = \left(\sum_{j=0}^{\infty} \mathfrak{A}_j(t)\epsilon^j\right) z, \tag{2.4}$$

the most general asymptotic expansions of solutions of this equation have been given by **H. L.** Turrittin in [9]. In particular, he has given sufficient conditions for the existence of a transformation $z = H(t, \epsilon)w$, where

$$H(t, \epsilon) = \sum_{k=0}^{K} \epsilon^k H_k(t), \tag{2.5}$$

K a suitable positive integer, which will transform equation (2.4) into

$$\epsilon^{\beta}\frac{dw}{dt} = \{(\delta_{ij}\lambda_j(t,\epsilon)) + \epsilon^{\beta}B(t,\epsilon)\}\,w(t,\epsilon)\,, \tag{2.6}$$

where the elements of $B(t, \epsilon)$ are $O(1)$ and the characteristic polynomials $\lambda_j(t, \epsilon)$ have the form

$$\lambda_j(t,\epsilon) = \sum_{k=0}^{\beta-1}\epsilon^k\lambda_{jk}(t), \qquad j = 1, 2, \cdots, N, \tag{2.7}$$

and $\lambda_j(t, \epsilon) \equiv \lambda_k(t, \epsilon)$, or $\lambda_j(t, \epsilon) \not\equiv \lambda_k(t, \epsilon)$, $a \le t \le b$, $0 < \epsilon \le \epsilon_0$. (Actually fractional powers of ϵ may be introduced; but an introduction of a new parameter could be made in the beginning, so that without loss of generality we assume that no fractional powers of ϵ occur.)

Further, if the characteristic polynomials $\lambda_j(t, \epsilon)$ are such that

$$\operatorname{Re}\{\epsilon^{-\beta}\lambda_1(t,\epsilon)\} \le \cdots \le \operatorname{Re}\{\epsilon^{-\beta}\lambda_N(t,\epsilon)\}, \qquad \text{for } a \le t \le b, \qquad 0 < \epsilon \le \epsilon_0,$$

there exists a fundamental matrix solution of (2.6) of the form

$$W(t,\epsilon) = F(t,\epsilon)E(t,\epsilon),$$

$$F(t,\epsilon) = \begin{pmatrix} F_{11} & \cdots & F_{1M} \\ \vdots & & \vdots \\ F_{M1} & \cdots & F_{MM} \end{pmatrix}; \qquad E(t,\epsilon) = \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & \cdot \\ \vdots & \vdots & \ddots & \cdot \\ 0 & & \cdots & E_M \end{bmatrix};$$

and asymptotically,

$$F_{ij} \sim \epsilon^{\beta ij}\sum_{k=0}^{\infty}F_{ijk}(t)\epsilon^k; \qquad \beta_{ij} > 0 \quad \text{if} \quad i \ne j; \qquad \beta_{ii} = 0;$$

$$E_i = I_i \exp\left\{\epsilon^{-\beta}\int_a^t \lambda_{r_i}(\sigma,\epsilon)\,d\sigma\right\},$$

where I_i is an identity matrix, and λ_{r_i} are the distinct characteristic polynomials.

If Turrittin's results apply to the individual equations

$$\epsilon^{h_j}\frac{d}{dt}y_j(t,\epsilon) = C_{jj}(t,\epsilon)y_j(t,\epsilon), \qquad j = 2, \cdots, p, \tag{2.8}$$

and $H_j(t, \epsilon)$ is the corresponding transformation required for Turrittin's canonical form, the transformation $x = Hz$, where

$$H(t,\epsilon) = T(t,\epsilon)\begin{bmatrix} H_1 & 0 & \cdots & 0 \\ 0 & H_2 & & \cdot \\ \vdots & & \ddots & \cdot \\ 0 & & \cdots & H_p \end{bmatrix}; \qquad H_1 \equiv I_r,\ T \text{ as in (2.1)}, \tag{2.9}$$

will change (1.1) into

$$\text{(2.10)}\qquad \begin{aligned} \frac{dz_1}{dt} &= D_{11} z_1 + D_{12} z_2 \\ \epsilon^h \frac{dz_2}{dt} &= D_{21} z_1 + D_{22} z_2, \end{aligned}$$

where L in (2.1) has been chosen so that $D_{12}(t, \epsilon) = O(\epsilon)$, $D_{21}(t, \epsilon) = O(\epsilon)$, $D_{22} = (\delta_{ij}\lambda_j(t, \epsilon)) + \epsilon^h \bar{D}_{22}$, $\bar{D}_{22}(t, \epsilon) = O(1)$ and $\lambda_j(t, \epsilon)$, $j = 1, \cdots, m$, are the non-vanishing characteristic polynomials associated with the differential systems (2.8).

It is advantageous to make one more transformation on (2.10), namely

$$\text{(2.11)}\qquad z(t, \epsilon) = P(t)u(t, \epsilon),$$

where $P(t)$ can be determined so that the new differential system is

$$\text{(2.12)}\qquad \begin{aligned} \frac{du_1}{dt}(t, \epsilon) &= B_{11}(t, \epsilon)u_1(t, \epsilon) + B_{12}(t, \epsilon)u_2(t, \epsilon) \\ \epsilon^h \frac{du_2}{dt}(t, \epsilon) &= B_{21}(t, \epsilon)u_1(t, \epsilon) + B_{22}(t, \epsilon)u_2(t, \epsilon), \end{aligned}$$

with related canonical degenerate differential system

$$\text{(2.13)}\qquad \begin{aligned} \frac{d}{dt} u_1 &= 0 \\ 0 &= (\delta_{ij}\alpha_j(t))u_2, \qquad \text{where } \alpha_j(t) \neq 0, \qquad a \leq t \leq b, \end{aligned}$$

such that the fundamental matrix solution $W(t, \epsilon)$ for (2.12) when Turrittin's results apply, has the form

$$\text{(2.14)}\qquad W(t, \epsilon) = ([I])E(t, \epsilon)^*$$

The effect of the transformations on the boundary form (1.2) will be considered in section 4.

3. The canonical problem

We make the following hypothesis.

H1: (i) *The matrices* $A_{ij}(t, \epsilon)$ *indicated in* (1.1) *have asymptotic expansions of appropriate high finite orders.*

(ii) *The matrices* $B_{ii}(t)$ *referred to in section* 2 *are non-singular* $a \leq t \leq b$.

(iii) *There exists a non-singular transformation* $x(t, \epsilon) = \bar{H}(t, \epsilon)u(t, \epsilon)$, *where* $\bar{H}(t, \epsilon) = \sum_{j=0}^{J} H_j(t)\epsilon^j$, $a \leq t \leq b$, $0 < \epsilon \leq \epsilon_0$, *which will convert* (1.1) *into the canonical form* (2.12).

(iv) *The* m *non-vanishing characteristic polynomials* $\lambda_j(t, \epsilon)$ *satisfy*

$$\operatorname{Re}\{\epsilon^{-h}\lambda_1(t, \epsilon)\} \leq \cdots \leq \operatorname{Re}\{\epsilon^{-h}\lambda_m(t, \epsilon)\}, \quad a \leq t \leq b, \quad 0 < \epsilon \leq \epsilon_0 .$$

* $[\phi(t)]$ represents a function $\phi(t,\epsilon) = \phi(t) + \epsilon^\gamma \phi_1(t,\epsilon)$, $\gamma > 0$, $|\phi_1(t,\epsilon)| < B$.

If $X(t, \epsilon)$ is any fundamental matrix for a system of differential equations of the form $dx/dt = A(t, \epsilon)x$, $\epsilon > 0$, $a \leq t \leq b$, then any particular vector solution $l(t, \epsilon)$ must be of the form $l = Xl(\epsilon)$. Thus if $l(t, \epsilon)$ is to satisfy the boundary conditions $R(\epsilon)l(a, \epsilon) + S(\epsilon)l(b, \epsilon) = c(\epsilon)$, we must have

$$\{RX(a, \epsilon) + SX(b, \epsilon)\}l(\epsilon) = c(\epsilon).$$

Thus, if $\Delta(\epsilon) = \{R(\epsilon)X(a, \epsilon) + S(\epsilon)X(b, \epsilon)\}$, $l(t, \epsilon)$ will be unique if $\Delta(\epsilon)$ is non-singular, for then

$$l(t, \epsilon) = X(t, \epsilon)\Delta^{-1}(\epsilon)c(\epsilon). \tag{3.1}$$

The limit problem is then the computation of the

$$\lim_{\epsilon\to 0+} l(t, \epsilon) = \lim_{\epsilon\to 0+} X(t, \epsilon)\Delta^{-1}(\epsilon)c(\epsilon).$$

To evaluate this limit we need more detailed information about the structure of $\Delta^{-1}(\epsilon)$.

4. $\Delta^{-1}(\epsilon)$ for the canonical problem

We assume that we are dealing with the canonical differential system (2.12) which has been obtained from (1.1) by the transformation

$$x(t, \epsilon) = \bar{H}(t, \epsilon)u(t, \epsilon)$$

of H1-(iii). This transformation will change the boundary conditions from (1.2) into

$$M(\epsilon)u(a, \epsilon) + N(\epsilon)u(b, \epsilon) = c(\epsilon), \tag{4.1}$$

where $M(\epsilon) = R(\epsilon)\bar{H}(a, \epsilon)$, and $N(\epsilon) = S(\epsilon)\bar{H}(b, \epsilon)$.

We make the following hypothesis.

H2: (i) $R(\epsilon) = R_0 + \epsilon R_1(\epsilon)$, $S(\epsilon) = S_0 + \epsilon S_1(\epsilon)$, *where the elements of* $R_1(\epsilon)$ *and* $S_1(\epsilon)$ *are bounded for* $0 \leq \epsilon \leq \epsilon_0$ *and the rank of* $(R(\epsilon) : S(\epsilon)) = n_1 + m$, $0 \leq \epsilon \leq \epsilon_0$.

(ii) *The non-vanishing characteristic polynomials have non-zero real parts,*

$$\operatorname{Re}\{\epsilon^{-h}\lambda_1(t, \epsilon)\} \leq \cdots \leq \operatorname{Re}\{\lambda^{-h}\lambda_k(t, \epsilon)\} < 0 < \operatorname{Re}\{\epsilon^{-h}\lambda_{k+1}(t, \epsilon)\} \leq \cdots \leq \operatorname{Re}\{\epsilon^{-h}\lambda_m(t, \epsilon)\},$$

$a \leq t \leq b$, $0 < \epsilon \leq \epsilon_0$.

If we choose for the fundamental matrix the one indicated in (2.14), we have

$$\Delta(\epsilon) = \left\{M(\epsilon)([I]) + N(\epsilon)([I])\begin{pmatrix} I_{n_1} & 0 \\ 0 & E_m(b, \epsilon)\end{pmatrix}\right\}.$$

If $D(\epsilon) = \det \Delta(\epsilon) \neq 0$, $\Delta(\epsilon)$ will be non-singular. We have

$$D(\epsilon) \backsim \textstyle\sum_\alpha \mathfrak{A}_\alpha(\epsilon)e^{w_\alpha(\epsilon)},$$

where

(i) α covers some finite range,

(ii) $w_\alpha(\epsilon)$ are distinct quantities, each of which is of the form

$$\sum_{j=I}^{J} \rho_{k_j}(b, \epsilon),$$

where $I, J = 0, 1, \cdots, m$; $J \geq I$, $(k_0, k_1, \cdots, k_m)$ is any permutation of $(0, 1, \cdots, m)$; $\rho_0(b, \epsilon) \equiv 0$, $\rho_j(b, \epsilon) = \epsilon^{-h}\int_a^b \lambda_j(\sigma, \epsilon)\, d\sigma$,

(iii) the coefficient functions $\mathcal{A}_\alpha(\epsilon) \not\equiv 0$, $\mathcal{A}_\alpha(\epsilon) = O(1)$ as $\epsilon \to 0+$.

For a discussion of the zeros of such exponential sums see Turrittin, ([10]).

All terms indicated in (ii) need not be present, however one is of particular interest, namely the term $\mathcal{A}(\epsilon)e^{w(\epsilon)}$ where $w(\epsilon) = \sum_{j=k+1}^{m} \rho_j(b, \epsilon)$. (We note that if $k = m$, then $w(\epsilon) \equiv 0$.) An explicit expression for the leading term of the coefficient function $\mathcal{A}(\epsilon)$ can be given as follows. Let the columns of $R_0\bar{H}(a, \epsilon)$ and $S_0\bar{H}(b, \epsilon)$ be α_{i1}, α_{i2} respectively, and let $\Omega(\epsilon)$ be the $n_1 + m$th order square matrix

$$\Omega = (\alpha_{11} + \alpha_{12} : \cdots \alpha_{n_1,1} + \alpha_{n_1,2} : \alpha_{n_1+1,1} : \cdots : \alpha_{n_1+k,1} : \alpha_{n_1+k+1,2} : \cdots \alpha_{n_1+m,2}). \tag{4.2}$$

The leading term of $\mathcal{A}(\epsilon)$ is the determinant of $\Omega(\epsilon)$. In general for ϵ sufficiently small it can be shown that if $\Omega(\epsilon)$ is non-singular (see Harris [5], page 88)

$$\Delta(\epsilon) = ([\Omega])\begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & E_{m-k}(b, \epsilon) \end{pmatrix},$$

where

$$E_{m-k}(b, \epsilon) = (\delta_{ij} \exp\{\rho_{k+j}(b, \epsilon)\}), \qquad j = 1, 2, \cdots, m - k,$$

and hence

$$\Delta^{-1}(\epsilon) = \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & E_{m-k}^{-1}(b, \epsilon) \end{pmatrix}([\Omega^{-1}(\epsilon)]). \tag{4.3}$$

It will be shown that the computation of the limit (3.1) as $\epsilon \to 0+$ is essentially the limit as $\epsilon \to 0+$ of the first n_1 rows of $\Omega^{-1}(\epsilon)$. To establish the nature of the elements in the first n_1 rows of $\Omega^{-1}(\epsilon)$ a detailed analysis of the transformation $\bar{H}(t, \epsilon)$ can be made (see Harris [7]) where it is shown that in general the elements in the first n_1 rows of $\Omega^{-1}(\epsilon)$ are $0(1)$.

We make the hypothesis

H3: *The matrix* $\Omega(\epsilon)$ *as given in* (4.2) *satisfies*

(i) $\Omega(\epsilon)$ *is non-singular,* $0 < \epsilon \leq \epsilon_0$,

(ii) *the elements in the first* n_1 *rows of* $\Omega^{-1}(\epsilon)$ *are* $O(1)$ *as* $\epsilon \to 0+$.

Hypothesis H3-(i) assures us that $\Delta^{-1}(\epsilon)$ has the form shown in (4.3).

5. Evaluation of $\lim_{\epsilon\to 0+} X(t, \epsilon)\Delta^{-1}(\epsilon)c(\epsilon)$

Combining (2.14) and (4.3) we have the following representation of the unique solution of (2.12) and (4.1).

$$l(t, \epsilon) = X(t, \epsilon)\Delta^{-1}(\epsilon)c(\epsilon) = ([I])\begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & E_k(t, \epsilon) & 0 \\ 0 & 0 & E_{m-k}(t, \epsilon)E_{m-k}^{-1}(b, \epsilon) \end{pmatrix}([\Omega^{-1}(\epsilon)])c(\epsilon).$$

We have

$$E_k(t, \epsilon) \to 0_k \quad a < \delta_1 \le t \le b,$$

$$E_{m-k}(t, \epsilon)E_{m-k}^{-1}(b, \epsilon) \to 0_{m-k} \quad a \le t \le \delta_2 < b,$$

exponentially fast due to H2-(ii), and uniformly in t for the indicated intervals. Thus

$$\text{(5.1)} \quad \lim_{\epsilon\to 0+} X(t, \epsilon)\Delta^{-1}(\epsilon)c(\epsilon) = \begin{pmatrix} l_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{\Omega}_{11}(0)c_1(0) + \bar{\Omega}_{12}(0)c_2(0) \\ 0 \end{pmatrix} = l,$$

$a < \delta_1 \le t \le \delta_2 < b$.

It is clear that the limiting constant vector l is defined for the interval

$$a \le t \le b$$

and as a function of t is a solution of the degenerate differential system (2.13). We shall now show that this limiting solution satisfies n_1 degenerate boundary conditions.

6. Boundary conditions satisfied by the limiting solution

Multiplication of the boundary form (4.1) on the left by any non-singular matrix of constants will give rise to an equivalent boundary form.

We have partitioned $\Omega(\epsilon)$ and $\Omega^{-1}(\epsilon)$ as follows

$$\Omega(\epsilon) = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}, \qquad \Omega^{-1}(\epsilon) = \begin{pmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} \\ \bar{\Omega}_{21} & \bar{\Omega}_{22} \end{pmatrix}.$$

We note that $M_{11}(0) + N_{11}(0) = \Omega_{1s}(0)$, $M_{21}(0) + N_{21}(0) = \Omega_{21}(0)$, and $\bar{\Omega}_{11}(\epsilon)\Omega_{11}(\epsilon) + \bar{\Omega}_{12}(\epsilon)\Omega_{21}(\epsilon) = I_{n_1}$ together with the nature of $\Omega_{11}(0)$, $\Omega_{21}(0)$ and H3-(ii) imply $\bar{\Omega}_{11}(0)\Omega_{11}(0) + \bar{\Omega}_{12}(0)\Omega_{21}(0) = I_{n_1}$ and the matrix $(\bar{\Omega}_{11}(0) : \bar{\Omega}_{12}(0))$ has rank n_1.

Thus, we have

$$\text{(6.1)} \qquad \bar{\Omega}_{11}(0)(M_{11}(0) + N_{11}(0)) + \bar{\Omega}_{12}(0)(M_{21}(0) + N_{21}(0)) = I_{n_1}.$$

Further, there exists constant matrices F_{21} and F_{22} such that the matrix

$$\text{(6.2)} \qquad F = \begin{pmatrix} \bar{\Omega}_{11}(0) & \bar{\Omega}_{12}(0) \\ F_{21} & F_{22} \end{pmatrix}$$

is non-singular. Let us replace (4.1) by the equivalent boundary form

$$\bar{M}(\epsilon)u(a, \epsilon) + \bar{N}(\epsilon)u(b, \epsilon) = \bar{c}(\epsilon), \tag{6.3}$$

where $\bar{M}(\epsilon) = FM(\epsilon)$, $\bar{N}(\epsilon) = FN(\epsilon)$, and $\bar{c}(\epsilon) = Fc(\epsilon)$.

The corresponding degenerate boundary form to (6.3) is

$$\bar{M}(0)u(a) + \bar{N}(0)u(b) = \bar{c}(0). \tag{6.4}$$

By direct computation and (6.1) we have

$$\bar{M}_{11}(0) + \bar{N}_{11}(0) = I_{n_1}$$

and

$$\bar{M}(0)\begin{pmatrix} l_1 \\ 0 \end{pmatrix} + \bar{N}(0)\begin{pmatrix} l_1 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 \\ (\bar{M}_{21}(0) + \bar{N}_{21}(0))l_1 \end{pmatrix}.$$

Also

$$\bar{c}(0) = \begin{pmatrix} \bar{c}_1(0) \\ \bar{c}_2(0) \end{pmatrix} = (F)c(0) = \begin{pmatrix} \bar{\Omega}_{11}(0) & \bar{\Omega}_{12}(0) \\ F_{21} & F_{12} \end{pmatrix}\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix},$$

so $\bar{c}_1(0) = \bar{\Omega}_{11}(0)c_1(0) + \bar{\Omega}_{12}(0)c_2(0) = l_1$.

Thus, the limiting solution l satisfies the first n_1 degenerate boundary conditions of (6.4) corresponding to the boundary form (6.3).

Without loss of generality we may assume that the boundary form (1.1) has been replaced by the equivalent one obtained by multiplication on the left by F as given in (6.2). Further, the solution of the canonical problem will provide the solution of the original problem thru the transformation of H1-(iii),

$$x(t, \epsilon) = \bar{H}(t, \epsilon)l(t, \epsilon). \tag{6.5}$$

The transformation $\bar{H}(t, \epsilon)$ was defined for $a \leq t \leq b$, $0 \leq \epsilon \leq \epsilon_0$, and $\lim_{\epsilon\to 0+} \bar{H}(t, \epsilon) = \bar{H}(t, 0)$ exists, $a \leq t \leq b$. Thus the limiting solution for the problem (1.1), (1.2) will be

$$x(t) = \bar{H}(t, 0)l, \qquad a < t < b. \tag{6.6}$$

THEOREM 1: *Under hypotheses* H1, H2, H3, *the two point boundary problem* (1.1), (1.2) *has a unique solution* $x(t, \epsilon)$ *on* $a \leq t \leq b$, $0 < \epsilon \leq \epsilon_0$, *such that the* $\lim_{\epsilon\to 0+} x(t, \epsilon) = x(t)$ *exists on the open interval* $a < t < b$, *and uniformly on any closed sub-interval. The function* $x(t)$ *satisfies the degenerate differential system* (1.3). *The limits* $x(a + 0)$ *and* $x(b - 0)$ *exist and satisfy the first* n_1 *boundary conditions of the degenerate boundary form* (1.4).

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