THE FORMAL THEORY OF SYSTEMS OF IRREGULAR HOMOGENEOUS LINEAR DIFFERENCE AND DIFFERENTIAL EQUATIONS*

By H. L. Turrittin

1. Introduction

The technique for computing formal power series solutions of the equations

$$\frac{dX}{ds} = A(s)X \tag{1}$$

and

$$X(s+1) = A(s)X(s), \tag{2}$$

where $X(s)$ and $A(s)$ are n by n square matrices and $A(s)$ has a convergent series representation

$$A(s) = s^{c/p} \sum_{k=0}^{\infty} A_k s^{-k/p}, \qquad |s| > s_0, \qquad A_0 \neq 0,$$

is roughly the same for the two equations. There are certain difficulties and complications however that arise in treating the difference equation (2) which do not appear in connection with the differential equation (1). A detailed procedure for computing formal solutions of the differential equation (1) was presented by H. L. Turrittin in 1955 ([1]). The earlier treatment in 1930 of the difference equation (2) given by G. D. Birkhoff ([2]) led to certain "indefinite algebraic complications", making Birkhoff's approach to the problem quite unsatisfactory from the computational point of view. In this paper a straight forward step-by-step process will be outlined for computing the formal series solutions of difference equations of type (2). The method about to be presented completely avoids Birkhoff's indefinite algebraic complications.

In the subsequent analysis a number of special cases will be first considered and then it will become evident how to obtain formal solutions in the most general case. We turn at once to the details of the process.

2. Formal solutions for cases I and II

Case I: If by chance (2) takes the special form

$$X(s+1) = \left(I + \sum_{k=p+1}^{\infty} A_k s^{-k/p}\right) X(s), \tag{3}$$

where I is the identity matrix, then a formal series solution

$$X(s) = I + \sum_{k=1}^{\infty} X_k s^{-k/p} \tag{4}$$

* This paper was prepared while working on the Office of Ordnance Research Project No. 1469, Contract No. DA-11-022-ORD-2042.

can be found at once, as one can easily verify by substituting series (4) directly into (3) and equating like powers of $s^{1/p}$ to determine $X_1, X_2, \cdots$ in succession. The series (4) thus determined will usually diverge.

Case II: If the order $n = 1$, both X and A are scalars. In this event G. D. Birkhoff in [2] has shown that equation (2) has a formal series solution of the form

$$X(s) = s^{\alpha_0 + cs/p} \exp\left\{\sum_{\nu=1}^{p-1} \alpha_\nu s^{\nu/p} + s \log \alpha_p\right\} \sum_{k=0}^{\infty} x_k s^{-k/p},$$

where $x_0 = 1$ and the constants $\alpha_p, \alpha_{p-1}, \cdots, \alpha_1, \alpha_0, x_1, x_2, x_3, \cdots$ can all readily be calculated, one after the other, in the order listed.

3. Separation of the characteristic roots of A_0

Returning to the general equation (2), when $n \geq 2$ and case I does not arise, we begin by reducing c to zero. To do so make the *power-of-s removing transformation*

$$X(s) = s^{cs/p} Y(s).$$

This transformation yields a new difference equation of exactly the same type as (2) for the unknown $Y(s)$, where, however, the c has been reduced to zero.

Once this has been done, as will be assumed from here on, a *canonicalizing transformation*

$$Y(s) = PZ(s), \tag{5}$$

where P is a constant non-singular matrix, is made to reduce the lead matrix A_0 to Jordan classical canonical form (hereafter abbreviated JCC-form). Thus when (2) is given there is no loss of generality in assuming it has the form

$$X(s+1) = \left(\sum_{k=0}^{\infty} A_k s^{-k/p}\right) X(s), \qquad A \not\equiv 0, \tag{6}$$

where A_0 has the special diagonal structure

$$A_0 = \begin{Vmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & M_m \end{Vmatrix} \quad \text{and} \quad M_i = \begin{Vmatrix} \rho_i & 0 & 0 & \cdots & 0 \\ \beta_i & \rho_i & 0 & & \vdots \\ 0 & \beta_i & \rho_i & & \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_i & \rho_i \end{Vmatrix} \tag{7}$$

$i = 1, \cdots, m$; β_i is either zero or unity; and the ρ_i's are the characteristic roots of A_0.

Let it be assumed for the moment that A_0 has the special structure

$$A_0 = \begin{Vmatrix} \mathfrak{M}_1 & 0 \\ 0 & \mathfrak{M}_2 \end{Vmatrix}, \tag{8}$$

where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are square matrices which have no common roots. Then in order to decompose equation (6) into two (or more) equations of lower order and, in so doing, separate the distinct roots of A_0, we shall make a sequence of *zero-inducing transformations* of the form

$$X(s) = [I + Q_k/(s-1)^{k/p}]Y(s), \tag{9}$$

where k is a positive integer and Q_k is a judiciously chosen constant matrix. As a result of such a transformation (9) one obtains a new equation

$$Y(s+1) = \left(\sum_{\nu=0}^{\infty} B_\nu s^{-\nu/p}\right) Y(s), \tag{10}$$

where, for example, $B_\nu = A_\nu$ for $\nu = 0, 1, \cdots, k-1$ and

$$B_k = A_k + A_0 Q_k - Q_k A_0 . \tag{11}$$

In particular, split matrix Q_k into four blocks of the same respective sizes as those in A_0, see (8), making the diagonal blocks zero, i.e. let

$$Q_k = \begin{Vmatrix} 0 & Q_{k12} \\ Q_{k21} & 0 \end{Vmatrix}$$

and likewise set

$$A_k = \begin{Vmatrix} A_{k11} & A_{k12} \\ A_{k21} & A_{k22} \end{Vmatrix} \quad \text{and} \quad B_k = \begin{Vmatrix} B_{k11} & B_{k12} \\ B_{k21} & B_{k22} \end{Vmatrix}$$

noting from (11) that in particular

$$\begin{cases} B_{k12} = \mathfrak{M}_1 Q_{k12} - Q_{k12}\mathfrak{M}_2 + A_{k12} \\ B_{k21} = \mathfrak{M}_2 Q_{k21} - Q_{k21}\mathfrak{M}_1 + A_{k21}. \end{cases}$$

Since it is known ([3]) that, if the two square matrices $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have distinct characteristic roots, there is one and only one matrix Q which satisfies the equation

$$\mathfrak{M}_1 Q - Q\mathfrak{M}_2 = A,$$

for any given A with the appropriate number of columns and rows, we begin by setting $k = 1$ and selecting Q_{112} and Q_{121} so that $B_{112} = 0$ and $B_{121} = 0$. As a result in the new equation (10) not only is B_0 in the diagonal form

$$B_0 = \begin{Vmatrix} \mathfrak{M}_1 & 0 \\ 0 & \mathfrak{M}_2 \end{Vmatrix}$$

but also B_1 is in a diagonal form

$$B_1 = \begin{Vmatrix} B_{111} & 0 \\ 0 & B_{122} \end{Vmatrix}$$

because of the choice of Q_1.

Next a second zero-inducing transformation is applied to (10) with $k = 2$ and by properly selecting Q_{212} and Q_{221} the first three coefficient matrices in the new equation can and will be diagonalized. At least formally one can repeat this process an infinite number of times first with $k = 1$, then $k = 2$, next $k = 3$, etc., finally obtaining an equation

$$(12) \qquad Z(s+1) = \left(\sum_{\nu=0}^{\infty} \begin{Vmatrix} \mathcal{C}_{\nu 11} & 0 \\ 0 & \mathcal{C}_{\nu 22} \end{Vmatrix} s^{-\nu/p}\right) Z(s),$$

where $\mathcal{C}_{011} = \mathfrak{M}_1$ and $\mathcal{C}_{022} = \mathfrak{M}_2$.

From the rigorous point of view the infinite product in the transformation

$$(13) \quad X(s) = \left[\left(I + \frac{Q_1}{(s-1)^{1/p}}\right)\left(I + \frac{Q_2}{(s-1)^{2/p}}\right)\left(I + \frac{Q_3}{(s-1)^{3/p}}\right)\cdots\right] Z(s),$$

which converts (6) into equation (12), is usually divergent; but this does not concern us here, for only formal manipulations are being carried out.

Equation (12) is entirely equivalent to the two separate equations

$$Z_{11}(s+1) = \left(\sum_{\nu=0}^{\infty} \mathcal{C}_{\nu 11} s^{-\nu/p}\right) Z_{11}(s)$$

and

$$Z_{22}(s+1) = \left(\sum_{\nu=0}^{\infty} \mathcal{C}_{\nu 22} s^{-\nu/p}\right) Z_{22}(s).$$

It is now clear that, *if in equation* (6) *two or more of the characteristic roots* ρ_i *of* A_0 *are distinct, system* (6) *can and will be decomposed into two or more separate systems of lower order, where the characteristic roots of the lead matrix in each system taken separately are all alike.*

In fact, if the roots ρ_i of A_0 were all distinct, $(i = 1, \cdots, n)$, equation (6) could be decomposed into n distinct scalar equations and by invoking case II a full set of formal solutions could be obtained.

4. Separation of the roots of A_r, $1 \leq r < p$

It is now evident that one may assume without loss of generality that the ρ_i in (7) are all alike. Therefore let $\rho_i = \rho$, $(i = 1, \cdots, m)$. Two distinct possibilities arise, either $\rho = 0$ or $\rho \neq 0$. Each of these two eventualities requires attention. Full details will be given for the case $\rho \neq 0$; and the situation which arises, if $\rho = 0$, can be treated in a like fashion, although for brevity these latter details will be omitted.

To begin with a simple situation, assume that in (6) $A_0 = \rho I$, where $\rho \neq 0$. If this be true make a *unitizing transformation*

$$X(s) = \rho^s Y(s)$$

and (6) will take the form (10) with $B_0 = I$.

To take into account the possibility at this stage that some of the coefficient matrices may be zero by chance or by virtue of a process of reduction, assume that the typical equation of order $n \geq 2$ under consideration has the form

$$X(s+1) = \left(I + \sum_{\nu=r}^{\infty} A_\nu s^{-\nu/p}\right) X(s), \qquad A_r \neq 0, \tag{14}$$

where $1 \leq r \leq p$.

If in (14) matrix A_r is not in JCC-form, a transformation of type (5) would produce the desired canonicalization; so also assume without loss of generality that in (14) the A_r is in the JCC-form.

Again the zero-inducing transformations of type (9) are once more brought into play, first with $k = 1$, then with $k = 2$, and so on. If $1 \leq r < p$, the final effect is to separate the distinct roots of A_r and reduce (14) to two (or more) equivalent systems if two (or more) roots of A_r are distinct. There is therefore no loss of generality in assuming the characteristic roots of A_r are all alike.

This separation of the roots of A_r can also be brought about when $r = p$, provided no two roots of A_p differ from one another by some integral multiple of $1/p$. It may well turn out that in separating the roots of A_r, when $r = p$, some or all of the resulting equations which arise from the separation of characteristic roots are scalar equations. In this event, in accordance with case II, one would obtain at once one or more of the desired formal solutions.

5. Further reductions of equation (14), cases III and IV

Once the roots of A_r in (14) have been separated in so far as possible as outlined above and matrix difference equations of order $n \geq 2$ still remain for consideration, they will again have structure (14); but this time only four possibilities remain:

Case III: The matrix $A_r = wI$, $w \neq 0$, and $r = 1, \cdots, p$.

Case IV: The $r = p$, and the characteristic roots of A_p are all alike; furthermore 1's are present on the subdiagonal of the canonical matrix A_p.

Case V: The $r = p$ and A_p has at least two distinct roots, and any two distinct roots of A_p differ by an integral multiple of $1/p$.

Case VI: The $r \geq 1$ and $r < p$ and all characteristic roots of A_r are equal, but 1's are present on the subdiagonal.

These four possibilities will be treated in the order listed.

Case III: If in (14) $A_r = wI$ and $w \neq 0$, the A_r matrix is annulled by making a *diagonal-element-removing transformation*

$$X(s) = \exp\left\{\frac{pws^{(p-r)/p}}{p-r}\right\} Y(s) \tag{15}$$

if $r = 1, 2, \cdots, p-1$ or

$$X(s) = s^w Y(s) \tag{16}$$

if $r = p$, and thus one obtains a new equation of the form

$$Y(s+1) = \left(I + \sum_{\nu=r+1}^{\infty} B_\nu s^{\nu/p}\right) Y(s). \tag{17}$$

Once equation (17) is reached and $r = p$, we are back to case I with its corresponding formal solution. If in (17) $r < p$, we again have an equation of type (14) with r *increased* a unit and once more a separation of the roots of A_{r+1} would be attempted. It is evident this process of separating roots or increasing the value of r will eventually reduce the equation or equations to problems of the type considered in cases I and II unless at some stage we reach an equation of one of the types labeled case IV, V, or VI.

Case IV: Here we are concerned with an equation with the structure

$$X(s+1) = \left(I + \sum_{\nu=p}^{\infty} A_\nu s^{-\nu/p}\right) X(s), \tag{18}$$

where $A_p = wI + E$ and E denotes an n by n constant matrix, $n \geq 2$, in which the first subdiagonal elements are either zeros or 1's, and at least one 1 is present. Furthermore all other elements in E off the first subdiagonal are zero.

Again a diagonal-element removing transformation (16) is applied to (18) to annul the w in A_p, in order to obtain a new equation

$$Y(s+1) = \left(I + \sum_{\nu=p}^{\infty} B_\nu s^{-\nu/p}\right) Y(s), \tag{19}$$

where $B_p = E$. Such an equation as (19) has a formal solution

$$Y(s) = \left(\sum_{j=0}^{\infty} Y_j s^{-j/p}\right) \exp\{E \log s\}, \tag{20}$$

where $Y_0 = I$ and $Y_1, Y_2, \cdots$ may be readily computed in succession by substituting (20) into (19) and equating coefficients of like powers of $s^{1/p}$ on both sides of the equation.

These remarks dispose of case IV and provide us with a way of obtaining a fundamental formal solution in the event that case IV should arise.

6. Reduction of case V to case IV or I

Case V: We are now ready to treat an equation

$$X(s+1) = \left(I + \sum_{\nu=p}^{\infty} A_\nu s^{-\nu/p}\right) X(s),$$

where A_p is in the diagonal block form

$$A_p = \begin{Vmatrix} \mathfrak{N}_1 & 0 & \cdots & 0 \\ 0 & \mathfrak{N}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathfrak{N}_m \end{Vmatrix}, \quad m \geq 2.$$

and $\mathfrak{N}_1, \cdots, \mathfrak{N}_m$ are all square matrices in JCC-form. Furthermore the roots of $\mathfrak{N}_i$, $(i = 1, \cdots, m)$ are all alike and equal to η_i. These η_i are the distinct characteristic roots of A_p and it may be assumed without loss of generality that the subscripts have been so ordered that $\eta_i = \eta_1 + \xi_i/p$, $(i = 1, \cdots, m)$, where the ξ_i are integers and

$$0 = \xi_1 < \xi_2 < \cdots < \xi_m .$$

Once more a sequence of zero-inducing transformations (9) with $k = 1$, $k = 2$, etc. is used to annul off diagonal blocks in so far as possible. The net effect, when the Q's are judiciously chosen, of the corresponding single transformation (13) is to produce a new equation of the form

$$Z(s+1) = (\| \delta_{ij} I_i \| + \| \delta_{ij} \mathfrak{N}_i \|/s + \| \mathfrak{C}_{ij}(s) \|) Z(s), \tag{21}$$

where $i, j = 1, \cdots, m$; δ_{ij} is the usual Kronecker delta; I_i is an identity matrix of the same order as $\mathfrak{N}_i$; and the block matrix $\mathfrak{C}_{ij}(s) \equiv 0$ if $i \geq j$ and

$$\mathfrak{C}_{ij}(s) = D_{ij} s^{(\xi_i - \xi_j - p)/p} \quad \text{if} \quad i < j,$$

where D_{ij} is an appropriate constant matrix.

Next apply to (21) a *root-equalizing transformation*

$$Z(s) = \| \delta_{ij} s^{\xi_i/p} I_i \| \, W(s) \tag{22}$$

to get a revised equation

$$W(s+1) = \left\{ \left\| \delta_{ij} \left(1 + \frac{1}{s}\right)^{-\xi_i/p} (I_i + \mathfrak{N}_i/s) + G_{ij}(s) \right\| \right\} W(s),$$

where $G_{ij}(s) \equiv 0$ if $i \geq j$ and

$$G_{ij}(s) = \left(1 + \frac{1}{s}\right)^{-\xi_i/p} D_{ij}/s \quad \text{if } i < j.$$

Hence (23) can be written in the form

$$W(s+1) = \left(I + \sum_{\nu=p}^{\infty} G_\nu s^{-\nu/p}\right) W(s),$$

where

$$G_p = \left\| \begin{matrix} \mathfrak{N}_1 - \xi_1 I_1/p, & D_{12} & , \cdots, & D_{1m} \\ 0 & , \mathfrak{N}_2 - \xi_2 I_2/p, & , & D_{2m} \\ \vdots & & \ddots, & D_{m-1,m} \\ 0 & \cdots & & 0, \mathfrak{N}_m - \xi_m I_m/p \end{matrix} \right\| .$$

Since $\mathfrak{N}_i$ is in the JCC-form and all the roots of $\mathfrak{N}_i$ are equal to $\eta_1 + \xi_i/p$, it is now obvious that all the characteristic roots of G_p are equal to η_1. Thus once G_p has been reduced to JCC-form, we are either back to case IV when 1's appear on the first sub-diagonal or, if no 1's appear, a diagonal-element-removing transformation of type (22) will bring us back to case I. This concludes the discussion of case V.

7. Treatment of case VI by a shearing transformation

Case VI: After a diagonal-element-removing transformation of type (15) has been employed, the typical equation under case VI will have the form

$$X(s+1) = \left(I + \sum_{\nu=r}^{\infty} A_\nu s^{-\nu/p}\right) X(s), \qquad 1 \leq r < p, \tag{24}$$

where

$$A_r = E = \| \delta_{ij} E_i \| \qquad i, j = 1, \cdots, m,$$

and either $E_1 = 0$ or every element in E_1 is zero except for 1's filling the first subdiagonal. Each square matrix $E_2, \cdots, E_m$ is made up of zero elements except for 1's filling the first subdiagonal. By hypothesis at least one matrix E_i is present with 1's on the subdiagonal. Further there is no loss of generality in assuming, as we do, that the E_i-matrices with 1's in them are arranged in order of size in E, the largest, if there is one, at the bottom.

Once again a sequence of zero-inducing transformations is applied, this time to equation (24), first with $k = 1$, $k = 2$, etc., and the $Q_1, Q_2, \cdots$ are judiciously chosen so as to obtain a new equation of the form

$$Z(s+1) = \left(I + \sum_{\nu=r}^{\infty} \mathfrak{C}_\nu s^{-\nu/p}\right) Z(s), \qquad 1 \leq r < p, \tag{25}$$

where

$$\mathfrak{C}_r = \| \delta_{ij} E_i \| \quad \text{and} \quad \mathfrak{C}_{r+k} = \| \mathfrak{C}_{r+k,ij} \| ;$$

$k = 1, 2, \cdots$ and $i, j = 1, \cdots, m$; and moreover every column in the block matrix $\mathfrak{C}_{r+k,ij}$ except the last (i.e. the right hand column) is filled with zero elements if $j \geq i$ and $j > 1$.

The block notation in (25) is now dropped and this equation (25) is rewritten so as to exhibit the individual elements in the matrices, many of which are zero. Thus (25) becomes

$$Z(s+1) = \left(I + \sum_{\nu=r}^{\infty} \| c_{\nu ij} \| \, s^{-\nu/p}\right) Z(s), \qquad 1 \leq r < p, \tag{26}$$

and $i, j = 1, \cdots, n$, where n denotes the order of matrix $Z(s)$. Equation (26) is now ready for a *shearing transformation*

$$Z(s) = \| \delta_{ij} s^{-\mu(n-i)/p} \| \, W(s), \qquad i, j = 1, \cdots, n, \tag{27}$$

which converts (26) into the new equation

$$W(s+1) = \left\| \delta_{ij}\left(1 + \frac{1}{s}\right)^{(n-i)/p} + \sum_{\nu=r}^{\infty} \left(1 + \frac{1}{s}\right)^{(n-i)\mu/p} c_{\nu ij} s^{[\mu(j-i)-\nu]/p} \right\| W(s). \tag{28}$$

We still must select the appropriate value for the positive rational number μ. First set $\mu = 1$, in (47) hoping that because a number of the elements $c_{\nu ij}$ happen to be zero, that when $\mu = 1$ equation (28) will take the form

$$W(s+1) = (I + \sum_{\nu=r+1}^{\infty} D_\nu s^{-\nu/p}) W(s), \tag{29}$$

where the D_ν are constant matrices. In this event we are back to an equation of type (14) *with r increased a unit* and the entire analysis so far described is reapplied to (29). Formal solutions are either obtained by the means described in cases I–V or we reach a new equation with r again one unit larger than in (29). Our process is then repeated several times, if necessary, until r is so large that we are back to case I and the procedure terminates with the computing of formal solutions as described in case I.

In order that equation (28) takes on form (29) when $\mu = 1$, it is necesary and sufficient in (28) that

$$c_{r+k,ij} = 0 \quad \text{when} \quad k = 1, \cdots, n-1 \quad \text{and} \quad j > i + k - 1. \tag{30}$$

The process for obtaining formal solutions thus far described breaks down if one or more of the constants in (30) are not zero. To handle even this eventuality consider the various sets of three subscripts $(r + k, i, j)$ which correspond to the c's listed in (30) *that are not zero* and using these and only these sets of $(r + k, i, j)$ values, pick the largest positive μ such that

$$r + k - \mu(j - i) \geq r + \mu.$$

Clearly this choice of μ which is to be used in (27), is rational and $0 < \mu < 1$.

Let this $\mu = \sigma/q$ where the integers σ and q are prime to one another. Then with this special choice of μ in (27) the equation (28) takes on the form

$$W(s+1) = (I + \sum_{\nu=rq+\sigma}^{\infty} F_\nu s^{-\nu/pq}) W(s), \tag{31}$$

where the F_ν are constant matrices.

This means we are back once again to an equation of type (14) with two new features; namely the fractional power of s has increased from p to pq and the constant lead matrix $F_{rq+\sigma}$ has a special structure. The entire reduction procedure so far outlined would be reapplied beginning with the reduction of $F_{rq+\sigma}$ to the JCC-form and the eventual splitting of equation (31) after a sequence of zero-inducing transformations into two or more equations of *lower order*, provided the characteristic roots of $F_{rq+\sigma}$ are not all alike. The process of reduction being outlined would then be applied to the lower ordered equations and always end up in the determination of formal series solutions provided that whenever an equation of type (31) was reached the roots of $F_{rq+\sigma}$ were not all equal.

If the roots of $F_{rq+\sigma}$ are all alike, it turns out, because of the special structure of $F_{rq+\sigma}$, they must then all be zero. A reapplication of the process here outlined will then result either in the splitting of the system into two or more equations of lower order or again the roots will all be alike and equal to zero, but this time *the number of* 1*'s on the first subdiagonal of the matrix corresponding to* $F_{rq+\sigma}$ *will increase*. Once the first subdiagonal has become full of 1's and the characteristic roots are still all zeros one more shearing transformation will either reduce the equation to case I or yield an equation which can be reduced to two or more equations of lower order. In any event the procedure will terminate and yield the desired formal solutions. For details substantiating the statements made in this paragraph compare the situation under consideration with that in reference [4].

8. Remaining cases VII and VIII

Two remaining cases need consideration.

Case VII: If by chance the lead matrix $A_0 = \rho I + E$, where $\rho \neq 0$, and E contains 1's on its subdiagonal, a shearing transformation of type (27) is made with $\mu = 1/n$. The effect of the transformation is to remove the 1's from the subdiagonal and make the new lead matrix corresponding to A_0 equal to ρI and thus the process described above becomes applicable.

Case VIII: In the event that $A_0 = E$ the steps proceed as in case VI beginning with a sequence of zero-inducing transformations, followed by a shearing transformation, and terminating with the determination of the formal series solutions, some of which this time may happen to be identically zero.

Summarizing then we see a straight forward step-by-step process for determining the formal series solutions of difference equations of type (2) is available as here indicated.

G. D. Birkhoff and Trjitzinsky ([5]) have shown that such formal series solutions represent true solutions of (2) asymptotically in the sense of Poincaré in appropriate sectors of the s-plane.

If one is interested in the possibility of summing such divergent series he may refer to the work of W. J. Trjitzinsky ([6]) and W. J. A. Culmer ([7]). As regards the summability of the formal series solutions of equations of both types (1) and (2), certain problems remain still unsolved.

University of Minnesota, Minneapolis

Bibliography

[1] H. L. Turrittin, *Convergent solutions of ordinary linear homogeneous differential equations in the neighborhood of an irregular singular point,* Acta Mathematica, 93 (1955) 27–66.

[2] G. D. Birkhoff, *Formal theory of irregular linear difference equations,* Acta Mathematica, 54 (1930) 205–246.

[3] C. E. Cullis, Matrices and determinoids, Vol. III. Pt. I, 419–421 and 441.

[4] H. L. Turrittin, *Asymptotic expansions of solutions of systems of ordinary linear differential equations containing a parameter,* Contributions to the theory of nonlinear oscillations, Vol. II, Annals of Math. Studies, No. 29, Princeton Univ. Press, 93–97.

[5] G. D. Birkhoff and W. J. Trjitzinsky, *Analytic theory of singular difference equations,* Acta Math., 60 (1933) 1–89.

[6] W. J. Trjitzinsky, *Laplace integrals and factorial series in the theory of linear differential and linear difference equations,* Trans. Amer. Math. Soc., 37 (1935), 80–146.

[7] W. J. A. Culmer, Convergent solutions of ordinary linear homogeneous difference equations in the neighborhood of an irregular singular point. Thesis, Univ. of Minn., Aug., 1959.