STABILITY OF PERIODIC SOLUTIONS OF GENERAL SECOND-ORDER EQUATIONS

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1. Basic equations

The equation considered is of the form

$$\ddot{x} + f(\dot{x}, x, T, t) = 0, \tag{1.1}$$

with f periodic in t, or with

$$f(t) = f(t + T). \tag{1.2}$$

The period is a parameter in the basic equation. We assume that a family of periodic solutions to (1.1) are known, parametrized by the real parameter ξ,

$$x = x(t, \xi) = x(t + T, \xi), \tag{1.3a}$$

$$T = T(\xi). \tag{1.3b}$$

Hereafter x refers to solutions of this family, except where used as subscripts to indicate differentiation. A proper parametrization is ensured by the conditions that T, x, $\dot{x}$, and $\ddot{x}$ be C^1 in ξ and that for no value of ξ is $x_\xi(t) \equiv 0$. The function f must be C^1 in its arguments.

The variational equation is now written, together with two closely related inhomogeneous equations:

$$\delta\ddot{x} + f_{\dot{x}}\delta\dot{x} + f_x\delta x = 0, \tag{1.4}$$

$$\dddot{x} + f_{\dot{x}}\ddot{x} + f_x\dot{x} = -f_t, \tag{1.5a}$$

$$\ddot{x}_\xi + f_{\dot{x}}\dot{x}_\xi + f_x x_\xi = -T_\xi f_T. \tag{1.5b}$$

The derivatives of f here are evaluated in terms of the known periodic solution. The quantities $f_{\dot{x}}$ and f_x are known periodic functions of t, so that (1.4) is a damped Hill's equation. For the other quantities in (1.5) we have the periodicity conditions

$$\dot{x}(t) = \dot{x}(t + T), \tag{1.6a}$$

$$x_\xi(t) = x_\xi(t + T) + T_\xi\dot{x}, \tag{1.6b}$$

$$f_t(t) = f_t(t + T), \tag{1.7a}$$

$$f_T(t) = f_T(t + T) + f_t. \tag{1.7b}$$

We now let

$$\mathfrak{F}(t) = \exp\left(\int_0 f_{\dot{x}}\, dt\right), \qquad \mathfrak{F}_0 = \mathfrak{F}(T), \tag{1.8}$$

with the integral with only a lower limit indicating an indefinite integral equal to zero at the lower limit. The quantity $\mathfrak{F}$ has the periodicity property

$$\mathfrak{F}(t + T) = \mathfrak{F}_0\mathfrak{F}(t). \tag{1.9}$$

The quantity $\ln \mathfrak{F}_0$ is termed the *average damping.* The two characteristic roots for the variational equation are termed a and b, and satisfy the condition

$$ab\mathfrak{F}_0 = 1. \tag{1.10}$$

The condition $\mathfrak{F}_0 \geqq 1$ is a necessary condition for stability.

We denote by u and v two suitably chosen independent solutions of the variational equation (1.4). We define Δ by the relation

$$\Delta = (u\dot{v} - v\dot{u})\mathfrak{F}, \tag{1.11}$$

and note that Δ is a constant. The solutions to (1.5) are then of the form

$$\dot{x} = \Delta^{-1}\left[u \int_0 \mathfrak{F}f_t v \, dt - v \int_0 \mathfrak{F}f_t u \, dt\right] + A_1 u + B_1 v, \tag{1.12a}$$

$$x_\xi = T_\xi \Delta^{-1}\left[u \int_0 \mathfrak{F}f_T v \, dt - v \int_0 \mathfrak{F}f_T u \, dt\right] + A_2 u + B_2 v. \tag{1.12b}$$

Our task is the characterization of the neutral points of the variational equation, in terms of *basic* types of neutral points. A set of basic types of neutral points is such that any neutral point, if not itself of a basic type, can be considered as a coalition of neutral points of basic types. One such basic type is obtained directly: A point on the ξ scale at which a and b are conjugate complex and at which $\mathfrak{F}_0 - 1$ or $\ln \mathfrak{F}_0$ has a simple zero is termed a *zero-damping neutral point.* In order to obviate inessential requirements on differentiability we may define a simple zero of a function as one across which the function changes sign. In the next section we consider the case of positive average damping, with $\mathfrak{F}_0(\xi) > 1$. In the third section we consider the special case of identically zero average damping, with $\mathfrak{F}_0(\xi) \equiv 1$.

2. Case of positive average damping

With positive average damping, the characteristic roots at and near any neutral point will be real and distinct, with $0 < ab < 1$. Within a stable region, the stability of solutions to (1.4) will be asymptotic. We choose $|a| > |b|$ for convenience, and we distinguish two distinct types of neutral points, one with $a = 1$ and the other with $a = -1$. An unstable region with $a > 1$ is termed of *positive type,* while an unstable region with $a < -1$ is termed of *negative type.* It is clear that two unstable regions of different types must be disjoint, with an intervening stable region.

A point on the ξ scale for which $1 + a$ has a simple zero (and $\mathfrak{F}_0 > 1$, of course) is termed an *antiperiodic neutral point.* Such points constitute a basic type of neutral point. Analogously, we may define a *periodic neutral point* corresponding to a simple zero of $1 - a$. A neutral point of either type may coalesce with a zero-damping neutral point, but a point of one type cannot coalesce with one of the other. However, we wish to characterize periodic neutral points further, to divide them into two basic subtypes.

We may choose u and v so that

$$\begin{pmatrix} u \\ v \end{pmatrix}_{(t+T)} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{(t)}. \tag{2.1}$$

The procedure now is to impose conditions (1.6) on the solutions (1.12), using the properties of (1.7) and (2.1). The analysis is straightforward and leads to the conditions

$$A_1(1 - a) = a\Delta^{-1} \int_0^T \mathfrak{F} f_t v \, dt, \tag{2.2a}$$

$$A_2(1 - a) = T_\xi \left[A_1 + a\Delta^{-1} \int_0^T \mathfrak{F} f_T v \, dt \right], \tag{2.2b}$$

together with the same conditions with a, A_1, and v replaced by b, B_1, and u. With the definition

$$G(\xi) = a\Delta^{-1} \int_0^T \mathfrak{F}[f_T + (1 - a)^{-1} f_t] v \, dt, \tag{2.3}$$

we may combine (2.2) to obtain

$$A_2(1 - a) = T_\xi G. \tag{2.4}$$

The arbitrariness in the definitions of A_2 and G as functions of ξ may be resolved by a suitable normalization of u.

First consider a point for which $T_\xi = 0$. Here $B_2 = 0$ (because $1 - b \neq 0$), and the conditions for a proper parametrization require that $A_2 \neq 0$. Hence at such a point $a = 1$ and the point is a periodic neutral point (or a coalition of such points). Such a point for which $G \neq 0$ and for which T_ξ has a simple zero is termed a *regular neutral point*.

If $T_\xi \neq 0$ it is possible for $G = 0$ without $a = 1$, provided $A_2 = 0$. A point for which $T_\xi \neq 0$, $A_2 \neq 0$, and for which G has a simple zero is termed an *anomalous neutral point*. Regular and anomalous neutral points are considered to be of different basic subtypes. Note that a coalition of a regular neutral point with an anomalous neutral point is a neutral point which does not divide stable and unstable regions on the ξ scale, as $1 - a$ does not have a simple zero there.

In characterizing possible neutral points we have *not* provided an algorithm for finding them, except for the obvious one that any zero of T_ξ is a neutral point with $\mathfrak{F}_0 \geqq 1$. The calculation of the function $G(\xi)$ essentially involves first the calculation of a and b, whence the stability properties would be automatically already known.

3. Case of zero average damping

An important special case is that in which the average damping is identically zero. This case is defined by $\mathfrak{F}_0(\xi) \equiv 1$ and $b = a^{-1}$. Stable solutions to (1.4) correspond to an a and a^{-1} conjugate complex, and are *not* asymptotically stable. Unstable solutions correspond to a real, with $a \neq 1$. Zero average damping appears when f in (1.1) is independent of $\dot{x}$ and may also appear when the dependence of f on $\dot{x}$ lies only in an additive term of the form $g(T, t)\dot{x}$, with

$$\int_0^T g \, dt \equiv 0.$$

We may best characterize the solutions in terms of a quantity α defined in terms of the characteristic roots by

$$2\alpha = a + a^{-1}. \tag{3.1}$$

If $\alpha > 1$ we are in an unstable region of positive type, and a simple zero of $\alpha - 1$ (at which $a = 1$) is termed a *periodic neutral point with zero damping.* If $\alpha < -1$ we are in an unstable region of negative type, and a simple zero of $\alpha + 1$ (at which $a = -1$) is termed an *antiperiodic neutral point with zero damping*. Our task is again to divide periodic neutral points into two basic subtypes.

The matrix formulation of (2.1) may not be used here in general, because at the point of interest the roots are not distinct. The exceptional case in which it might be used is generally termed degenerate, and a degenerate neutral point can best be considered as a coalition of two nondegenerate points. In considering the nondegenerate case we can stay in the domain of real analysis by using the rational canonical form for the matrix (see, for example, [1]) in the place of a form with the roots on the diagonal. Accordingly, we define u and v so that we have

$$\begin{pmatrix} u \\ v \end{pmatrix}_{(t+T)} = \begin{pmatrix} 0 & 1 \\ -1 & 2\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{(t)} \tag{3.2}$$

in place of (2.1).

We next follow the same procedure as in the previous section, using (3.2) in place of (2.1). Obtained from the analysis again are four conditions, which may be reduced to two of interest. One of these expresses $A_2 + B_2$ as a suitable multiple of T_ξ. The other may be expressed

$$2(\alpha - 1)B_2 = T_\xi \mathcal{G}, \tag{3.3}$$

where

$$\mathcal{G} = \Delta^{-1} \int_0^T \mathfrak{F}\{f_T[(2\alpha - 1)u - v] - f_t u\} \, dt. \tag{3.4}$$

If $T_\xi = 0$ at a point on the ξ scale, $A_2 + B_2 = 0$, the condition for proper parametrization gives $B_2 \neq 0$, and hence $\alpha = 1$. A point at which $\mathcal{G} \neq 0$ and T_ξ has a simple zero is termed a *regular neutral point with zero damping*.

If $T_\xi \neq 0$ and $\mathcal{G} = 0$ at a point it is possible for $\alpha \neq 1$, provided $B_2 = 0$ there. A point for which $T_\xi \neq 0$ and $B_2 \neq 0$ and for which $\mathcal{G}$ has a simple zero is termed an *anomalous neutral point with zero damping*. A neutral point which appears to be of this type has been found in the response curve for Duffing's equation by R. de Vogelaere (private communication).

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BIBLIOGRAPHY

[1] H. W. TURNBULL and A. C. AITKEN, An introduction to the theory of canonical matrices, corrected reprinting, Blackie, London and Glasgow, 1945.