ON UNSTABLE ATTRACTORS

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1. Introduction

The present paper grew out of a search (initiated by a query of R. Kalman) for an isolated critical point in some dynamical system $D = (R, I, f)$, which has the property that it is the (positive) limit of all motions in the space without being stable. It is easy to construct such critical points if the space R is compact (see, for instance, example 3 in [1]). But if R is not compact (e.g. if R is E^n, $n \geq 2$) no examples were known. However such critical points do exist and an example of one in E^2 is given in section 6. The example just referred to was not inspired by some pictorial image. Instead, it was systematically derived through an analysis of the necessary properties of the desired singularity. This analysis, which is presented here, makes essential use of a rather general theorem (see [1] and Theorem 2 in this paper) concerning necessary and sufficient conditions for the parallelizability of subsystems of given dynamical systems. For that reason the following discussion is interesting not only for the light it sheds on an unexpected singularity, but also for the application it affords to the above mentioned result.

2. Preliminaries

Let R be a locally compact, separable metric space[1] and let $D = (R, I, f)$ be a dynamical system[2] defined in R. For a point p in R, we let $f_p(t)$ and $f_p(I)$ denote the motion through p and its orbit[3], respectively. The positive [negative] semi-orbit through p is the set $f_p(I^+)$ $[f_p(I^-)]$[4]. The symbol $S(p, \eta)$ stands for the open sphere of radius η and center at p.

The following definitions are standard: A point q in R is an *ω-limit point* [*α-limit point*] of $f_p(t)$ if there exists an unbounded sequence of positive [negative]numbers t_n, $(n = 1, 2, \cdots)$, such that $f_p(t_n) \rightarrow q$ as $n \rightarrow +\infty$. The motion $f_p(t)$ is *$L^+[L^-]$-stable* if the closure of $f_p(I^+)[f_p(I^-)]$ is compact; it is *L-stable* if the closure of $f_p(I)$ is compact[5]. A point p is said to be *wandering* if there exists a $\delta > 0$ and a $T > 0$ such that the intersection $S(p, \delta) \cap f_t(S(p, \delta))$[6] is empty for all t satisfying $|t| > T$. The point $p \in R$ is a critical point if $f_p(t) = p$ for all $t \in T$. The system D has an *improper saddle point* if there exists a sequence

[1] The metric in R is designated by ρ.

[2] Here I represents the real line $-\infty < t < +\infty$ and f maps $R \times I$ into R in such a way that the triplet D is a one parameter group of transformations acting on the space R.

[3] The mapping f_p: $I \rightarrow R$ defined by $f_p(t) \rightarrow f(p, t)$, p fixed in R, $t \in I$, is called a *motion*, whereas the point set $f_p(I)$ is called a *path* or *orbit*.

[4] Here I^+, I^- denotes the sets $0 \leq t < +\infty$, $-\infty < t \leq 0$, respectively.

[5] This is the notion of stability in the sense of Lagrange.

[6] For every fixed t in I, the mapping $f_t : R \rightarrow R$ is the homeomorphism defined by $f_t(p) = f(p, t)$.

of points $\{p_n\}$ and two unbounded sequences $\{t_n\}$, $\{\tau_n\}$ of real numbers such that

$$0 < \tau_n < t_n, \qquad (n = 1, 2, \cdots),$$

$$p_n \to p, \qquad f(p_n, t_n) \to q \text{ as } n \to +\infty,$$

whereas the sequence $\{f(p_n, \tau_n)\}$ has no limit point. Finally, we recall the following two definitions:

Definition 1. A critical point O in R is said to be (*positively*) *stable* if given any ϵ, $\epsilon > 0$ there exists a δ, $0 < \delta < \epsilon$ such that all points q in $S(O, \delta)$ have the property that $f_q(I^+)$ is contained in $S(O, \epsilon)$.

Definition 2. A critical point O in R is said to be (*positively*) *asymptotically stable* if (i) it is stable and (ii) for some δ and for all q in $S(O, \delta)$, $f_q(t)$ tends to O as $t \to +\infty$.

3. Statement of problem

The problem, in its least pathological form, is the following: To find a critical point O in R (say $R = E^n$, $n \geq 2$) having the following properties:

(3.1)
(i) $\{O\}$ is the only minimal set[7] in D
(ii) For all p in R, $f_p(t) \to O$ as $t \to +\infty$
(iii) O is not stable.

Remark: Condition (3.1) (i) is equivalent to the requirement that every compact invariant set in D contain O.

4. Some necessary conditions

We shall assume henceforth that R is the n-dimensional Euclidean space E^n, $n \geq 2$. (The following discussion can be carried out for somewhat more general spaces, but to no apparent advantage). Suppose O is a critical point in R satisfying properties (3.1). Since O is unstable there exists an ϵ^*, $\epsilon^* > 0$, such that $f(S(O, \delta), I^+)$[8] is not contained in $S(O, \epsilon^*)$ for any δ, $0 < \delta < \epsilon^*$. This could happen either if:

(i) $f(S(O, \delta), I^+)$ is unbounded for all $\delta > 0$, or if
(ii) $f(S(O, \delta), I^+) \subset S(O, \eta)$, $\eta > \epsilon^*$, for some $\delta < \epsilon^*$ and some η.

The fact that case (i) is not possible is a corollary to the following Lemma (see Corollary 2).

Lemma 1. *Let D satisfy condition* (3.1) (ii). *If p_n tends to p as $n \to +\infty$ and $\{t_n\}$ is an arbitrary sequence of positive numbers, then the sequence $\{f(p_n, t_n)\}$ is bounded.*

[7] A set A is *invariant* if $f_t(A) \subset A$ for all $t \in I$. A set is *minimal* if it is compact, invariant and minimal relative to these properties.

[8] For any set A in R we define $f(A, I^+)$ to be the set

$$\{f(p, t) \mid p \in A, t \in I^+\}$$

PROOF. Assume the Lemma false. Then there exists a sequence of points $\{p_n\}$ and a sequence of positive numbers $\{t_n\}$ such that p_n tends to p as $n \to +\infty$, whereas $\{f(p_n, t_n)\}$, $(n = 1, 2, \cdots)$, is unbounded. The sequence $\{p_n\}$, being bounded, is contained in $S(O, N)$ for some sufficiently large integer N. On the other hand, since $\{f(p_n, t_n)\}$ is unbounded, we may assume, without loss of generality (w.l.o.g.), that $\rho(f(p_n, t_n), O) \geq N + 1$, $(n = 1, 2, \cdots)$.

Let the arc $\{f_p(t) | t_1 \leq t \leq t_2 ; t_1, t_2 \in I\}$ be denoted by $f(p; t_1, t_2)$ and the boundary of $S(O, N)$ by $\sum (O, N)$. Then $f(p_n; 0, t_n)$, $(n = 1, 2, \cdots)$, must intersect $\sum (O, N)$, and we designate the last such point of intersection by p_n'. Clearly p_n', $(n = 1, 2, \cdots)$, may be represented as the point $f(p_n, t_n')$ for some uniquely determined[9] t_n' satisfying $0 < t_n' < t_n$. For the sake of convenience we require, w.l.o.g., that p_n' tend to p' on $\sum (O, N)$.

It is a consequence of our construction that the arc $f(p_n'; 0, t_n - t_n')$, $(n = 1, 2, \cdots)$, does not intersect $S(O, N)$. We write $t_n - t_n' = \tau_n$, $(n = 1, 2, \cdots)$, in which case the points $f(p_n, t_n)$ and $f(p_n', \tau_n)$, $(n = 1, 2, \cdots)$, are identical.

It is easy to see that $\tau_n \to +\infty$ as $n \to +\infty$, for if $\{\tau_n\}$ contained a bounded subsequence, it would contain a convergent subsequence converging, say, to τ whence, by continuity, $\{f(p_n, t_n)\} = \{f(p_n', \tau_n)\}$, $(n = 1, 2, \cdots)$, would have a limit point, namely $f(p', \tau)$, contrary to assumption.

Since $f(p', t)$ tends to O as $t \to +\infty$, we may choose T, $T > 0$, such that $f(p', T)$ is contained in $S(O, N/2)$. For the sake of convenience we take $N \geq 8$. Using continuity of f we choose an M such that $\rho(f(p_n', t), f(p', t)) < 1$ for all $n \geq M$ and all $0 \leq t \leq T$. Since $f(p'; 0, T)$ intersects $\sum (O, N/2)$ and since $N/4 > 1$, it follows that, for all $n \geq M$, the arc $f(p_n'; O, T)$ intersects $\sum (O, 3N/4)$. But if n is sufficiently large τ_n exceeds T and therefore the arc $f(p_n'; 0, T)$ is properly contained in $f(p_n'; 0, \tau_n)$. The latter arc, however, does not intersect $S(O, N)$. The contradiction just displayed completes the proof of Lemma 1.

COROLLARY 1. *Under the conditions of Lemma* 1, *if A is a bounded subset of R then $f(C(A), I^+)$*[10] *is bounded.*

PROOF. The proof is trivial

COROLLARY 2. *Under the same conditions for any δ, $\delta > 0$, there exists an $\eta = \eta(\delta)$, $0 < \delta < \eta$, such that $f(S(O, \delta), I^+)$ is contained in $S(O, \eta)$.*

COROLLARY 3. *If for all p in R, $f_p(t)$ tends to O as $t \to +\infty$ then D has no improper saddle points.*

PROOF. The proof follows directly from Lemma 1 and the definition of improper saddle points.

LEMMA 2. *Let D be a dynamical system without improper saddle points. Let p_n tend to p and $f(p_n, t_n)$ tend to q as $n \to +\infty$. Let A be the set $\bigcup_{n=1}^{\infty} f(p_n; 0, t_n)$. Then $C(A)$ is compact.*

[9] Uniqueness follows from the fact that the existence of periodic solutions would be incompatible with condition (3.1) (ii).

[10] $C(A)$ denotes the closure of A.

PROOF. For a proof see Lemma 4 in [1].

LEMMA 3. *Let D satisfy conditions* (3.1) (i), (ii). *If O is not an α-limit point of any motion other than itself, then $D^* = (R - O, I, f|R - O)$ has no improper saddle points.*

PROOF. Assume the contrary. Suppose p_n tends to p and $q_n = f(p_n, t_n)$ tends to q, where both p and q are points in $R - O$. Let $\{\tau_n\}$, $0 < \tau_n < t_n$, $(n = 1, 2, \cdots)$, be such that the sequence $\{f(p_n, \tau_n)\}$ has no limit point in $R - O$. But $\{f(p_n, \tau_n)\}$, $(n = 1, 2, \cdots)$, has at least one limit point in R (Lemma 1) hence this limit point must be the point O and $f(p_n, \tau_n)$ actually tends to O as $n \to +\infty$. We write $f(q_n; -\tau_n')$ for $f(p_n, \tau_n)$ where $\tau_n' = t_n - \tau_n$, $(n = 1, 2, \cdots)$, and observe that an argument similar to one used in the proof of Lemma 1 yields that $\tau_n' \to +\infty$ as $n \to +\infty$. We now have:

(i) $q_n \to q$ as $n \to +\infty$, $q \in R - O$

(ii) $f(q_n, -\tau_n') \to O$, $\tau_n' \to +\infty$ as $n \to +\infty$

Let $A = \bigcup_{n=1}^{\infty} f(q_n; 0, -\tau_n')$. Then $C(A)$ is compact (Lemma 2). If $t \in I^-$ then $f(q_n, t)$ tends to $f(q, t)$ as $n \to +\infty$. Furthermore $f(q_n, t)$ is contained in $f(q_n; 0, -\tau_n')$ for all n sufficiently large. Thus $f(q, t) \in C(A)$ for all $t \in I^-$ whence $f_q(I^-) \subset C(A)$, a compact set. The motion $f_q(t)$ is therefore L^--stable and its α-limit set A_q must contain a minimal set. Since the only minimal set in D is the set consisting of the isolated critical point O, it follows that O is in A_q, contradicting our original assumption. This completes the proof of Lemma 3.

Suppose $\{O\}$ is the only minimal set in D and $f_p(t) \to O$ as $t \to +\infty$ for all p in R. Then no motion in $D^* = (R - O, I, f|R - O)$ is either L^+ or L^--stable. For otherwise the α- or ω-limit set of that motion would be a compact invariant set contained in $R - O$. But every compact invariant set must contain a (non-empty) minimal set. The system D^* consists, therefore, solely of motions which are L-unstable.

We recall that in a dynamical system without improper saddle points every point which is L-unstable is wandering (see [1], Theorem 2 and [2], §9). It follows from Lemma 3 that if O is not an α-limit point of any motion other than itself then all points in $R - O$ are wandering.

Niemyckii and Stepanov showed ([2], §9) that for a dynamical system to be equivalent to a system of parallel lines in Hilbert space, (or in E^{n+1} if R is simply the n dimensional Euclidean space E^n)[11] it is necessary and sufficient that the system have no improper saddle point and that every point be wandering.

We have therefore completed the proof of the following Lemma:

LEMMA 4. *Let $\{O\}$ be the only minimal set in D and let $f_p(t) \to O$ as $t \to +\infty$ for all p in R. Suppose furthermore that O is not an α-limit point of any motion other than itself. Then $D^* = (R - O, I, f|R - O)$ is parallelizable.*

[11] D is said to be *equivalent to a system of parallel lines in Hilbert space H*, or simply *parallelizable*, if there exists a homeomorphism of the space R onto a subset of H which is order preserving on the orbits of D and carries these orbits onto a family of parallel lines in H.

THEOREM 1. *If* $\{O\}$, *is the only minimal set in* D *and* $f_p(t) \to O$ *as* $t \to +\infty$ *for all* p *in* R, *and if, furthermore,* O *is not an* α*-limit point of any motion other than itself then* O *is (positively) asymptotically stable.*

PROOF. Theorem 1 is an easy consequence of the following general result proved by the author in [1].

THEOREM 2. *Let* $D = (R, I, f)$ *be an arbitrary dynamical system and suppose* C, $C \subset R$, *is a compact invariant set (of the system* D*), the connected components of which are designated by* $C^1, \cdots, C^h$. *Assume that* $R - C$ *is connected*[12] *and denote the system* $(R - C, I, f|R - C)$ by D^*.

The following is a set of necessary and sufficient conditions that D^* *be parallelizable:*

I. *The system* D *has no improper saddle points.*

II. *Given any* ϵ, $\epsilon > 0$, *there exists a* δ, $0 < \delta < \epsilon$, *such that any motion starting within the compact* δ *neighborhood of* C^i, $(i = 1, \cdots, h)$, *(notation* W_δ^i*), has at least one semi-orbit in* W_ϵ^i.

III. (1) *If* R *is not compact then* C *is connected and every motion in* D^* *tends to* C *as* $t \to +\infty$ $(t \to -\infty)$. (2) *If* R *is compact, then either*:

2.1. C *is connected and every motion in* D^* *tends to* C *in both directions*; or

2.2. C *has exactly two components and every motion in* D^* *tends to one of these components when* $t \to +\infty$ *and to the other when* $t \to -\infty$.

It is a consequence of Lemma 4 that the "necessary" part of Theorem 2 applies with the set C consisting of the single critical point O. It follows that given any ϵ, $\epsilon > 0$, there exists a δ, $0 < \delta < \epsilon$, such that any motion starting within $S(O, \delta)$ has at least one semi-orbit contained in $S(O, \epsilon)$. The assumptions of Theorem 1 clearly preclude the possibility that the negative semi-orbit of any motion (other than O) is contained in $S(O, \epsilon)$. Therefore, given any ϵ, $\epsilon > 0$, there exists a δ, $0 < \delta < \epsilon$, such that any motion starting within $S(O, \delta)$ has its positive semi-orbit contained in $S(O, \epsilon)$. The critical point O is thus (positively) stable. The fact that its stability is asymptotic is inherent in our original assumptions. This completes the proof of Theorem 1.

5. Some more necessary conditions

We have shown thus far that an unstable attractor (and by that is meant a critical point having properties (3.1)) must satisfy the following necessary conditions:

$$(5.1)\qquad \begin{aligned} &\text{(i) For every } \delta,\ \delta > 0,\ f(S(O, \delta), I^+) \text{ is bounded.} \\ &\text{(ii) } O \text{ is an } \alpha\text{-limit point of motions other than itself.} \end{aligned}$$

Suppose O is a critical point of the desired kind. Let N denote the (invariant) set of all points in R the motions through which have O in their α-limit set. If

[12] For a more general statement of the theorem covering the case when R-C has an arbitrary number of components see Theorem 3 in [1].

q is in N then given any δ, $\delta > 0$, there is a t^* large enough so that $q^* = f(q, -t^*)$ is contained in $S(O, \delta)$. But then $q = f(q^*, t^*)$ and therefore q is contained in $f(S(O, \delta), I^+)$, which is bounded. Hence N is bounded.

LEMMA 5. *The set N is closed, hence compact.*

PROOF. Let q_n, $(n = 1, 2, \cdots)$, be a sequence of points in N which tends to the point q in R. Choose a sequence of numbers τ_n such that $\tau_n \geq n$, $(n = 1, 2, \cdots)$, and $f(q_n, -\tau_n) \to O$ as $n \to +\infty$. Such a choice of τ_n is clearly possible by virtue of the definition of N. Let A be the set $\bigcup_{n=1}^{\infty} f(q_n; 0, -\tau_n)$. Then $C(A)$ is compact (Lemma 2) and $f_q(I^-)$ is contained in $C(A)$. But if the motion $f_q(I)$ is L^--stable, its α-limit set contains a non-empty minimal set, that is, it must contain O. Hence q is in N and the proof of Lemma 5 is complete.

Consider the system $(R - N, I, f|R - N)$, which we denote by D^*. Every point in $R - N$ is clearly L-unstable (otherwise condition (3.1) (i) would be violated, since O is contained in N). We wish to show that D^* is parallelizable. Use will be made of the following Lemma:

LEMMA 6. $D^* = (R - N, I, f|R - N)$ *has no improper saddle points.*

PROOF. Assume the contrary. Then there exists a sequence of points $\{p_n\}$ and sequences of real numbers $\{t_n\}$, $\{\tau_n\}$ such that

(i) $p_n \in R - N$, $(n = 0, 1, 2, \cdots)$, $p_n \to p_0$ as $n \to +\infty$

(ii) $0 < \tau_n < t_n$, $(n = 1, 2, \cdots)$,

(iii) $f(p_n, t_n) \to q$ as $n \to +\infty$, $q \in R - N$

(iv) $\{f(p_n, \tau_n)\}$ has no limit points in $R - N$.

Since $\{f(p_n, \tau_n)\}$ is bounded (Lemma 1), it has a limit point p^* in R, and we assume, w.l.o.g., that $f(p_n, \tau_n) \to p^*$ as $n \to +\infty$. Clearly p^* is in N.

We write p_n^1 for $f(p_n, \tau_n)$ and τ_n^1 for $t_n - \tau_n$, $(n = 1, 2, \cdots)$. Then $f(p_n, t_n) = f(p_n^1, \tau_n^1)$, $(n = 1, 2, \cdots)$, and it is clear (since N is invariant) that $\tau_n^1 \to +\infty$ as $n \to +\infty$. Let A be the set $\bigcup_{n=1}^{\infty} f(p_n^1; 0, \tau_n^1)$. Then $C(A)$ is compact (Corollary 3 and Lemma 2) and $f_q(I^-)$ is contained in $C(A)$. Hence O is in the α-limit set of $f_q(t)$, whence q is in N, which is absurd. This completes the proof of Lemma 6.

COROLLARY 4. $D^* = (R - N, I, f|R - N)$ *is parallelizable.*

PROOF. Since all points in $R - N$ are L-unstable, it follows from Lemma 6 that all points in $R - N$ are wandering. Therefore, by the theorem of Niemyckii-Stepanov D^* is parallelizable.

We are now able to apply theorem 2 quoted above to the system D^*. A simple argument, similar to the one used in the proof of Theorem 1 yields the fact that the set N is (positively) asymptotically stable with respect to $R - N$[13]. We summarize the situation in the following Theorem.

THEOREM 3. *Let O be a critical point in $D = (R, I, f)$ having the following properties*: (i) $\{O\}$ *is the only minimal set in D*, (ii) *for all $p \in R$, $f_p(t) \to O$ as*

$t \to +\infty$, and (iii) *O is not (positively) stable. We use N to denote the "nodal region" in R, consisting of all those points in R, which have O in both their α- and ω-limit sets. Then O has the following properties:*

(i) *For any $\delta > 0$, $f(S(O, \delta), I^+) \subset S(O, \eta)$ for some $\eta = \eta(\delta)$.*

(ii) *$N - O$ is not empty; N is a compact invariant set.*

(iii) *$D^* = (R - N, I, f|R - N)$ is parallelizable, or, equivalently, N is (positively) asymptotically stable with respect to $R - N$.*

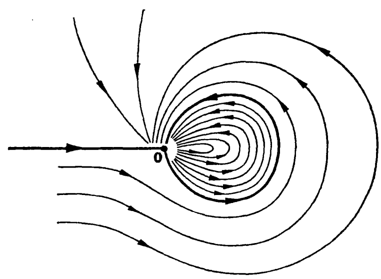

FIG. 1

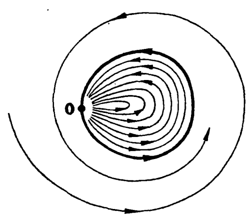

FIG. 2

6. Examples

The phase portrait of an unstable attractor in E^2 is given in Figure 1. This phase portrait is completely suggested by Theorem 3.

Remark: If we replace the requirement that for all p in R $f_p(t)$ tend to O as $t \to +\infty$ by the weaker requirement that for all p in $Rf_p(t)$ have O as an ω-limit point, the discussion carried out above goes through with insignificant modifications. An example of this less stringent critical point in E^2 is offered in figure 2.

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[2] V. NIEMYCKII, and V. STEPANOV, Qualitative theory of differential equations, Chapter IV, translation to appear, Princeton University Press.

[13] An invariant set N is said to be *(positively) asymptotically stable* with respect to an invariant set M if given any ϵ, $\epsilon > 0$ there exists a δ, $0 < \delta < \epsilon$, such that any motion emanating in the set M which intersects the δ neighborhood of N, stays in the ϵ neighborhood of N for all subsequent time.