STABILITY AND ASYMPTOTIC THEORY FOR LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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In this paper we shall describe some recent work in which we have extended, to systems of differential-difference equations, several familiar results in the theory of stability and asymptotic behavior of solutions of systems of ordinary differential equations. Reference will be made to more detailed expositions elsewhere ([3] and [4]).

1. Stability theory

In one aspect of the stability theory for ordinary linear differential equations, one considers a system of equations of the form

$$\frac{dy}{dt} = A(t)y, \tag{1}$$

where t is a real variable, y is an n-vector, and $A(t)$ is an n-by-n matrix, and a perturbed system

$$\frac{dz}{dt} = [A(t) + B(t)]z. \tag{2}$$

in which z is also an n-vector. In general terms, the stability problem is to determine conditions on the matrix $B(t)$ sufficient to insure that some property of all solutions of the equation in (1)—such as boundedness or order of growth—will also be a property of all solutions of the equation in (2). This stability problem has been extensively investigated for ordinary differential equations (cf. Bellman, [1]). In the first part of our work, we have considered, instead of the functional equations in (1) and (2), the systems of linear differential-difference equations[1]

$$y'(t + h_m) + \sum_{k=0}^{m} A_k(t)y(t + h_k) = 0 \tag{3}$$

and

$$z'(t + h_m) + \sum_{k=0}^{m} [A_k(t) + B_k(t)]z(t + h_k) = 0. \tag{4}$$

Here $A_k(t)$ and $B_k(t)$ ($k = 0, 1, \cdots, m$) represent given matrix functions, and y and z are n-vectors. The "spans" $h_0, h_1, \cdots, h_m$ are assumed to be real, and can be supposed to satisfy the conditions $0 = h_0 < h_1 < \cdots < h_m$. The stability property we consider is that of boundedness of all solutions as $t \to +\infty$.

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[1] For a survey of the general theory of differential-difference equations, refer to Bellman, [2].

One of the most striking features of our method is the use of the concept of the adjoint equation, rather than the more primitive concept of the inverse of a matrix which is customarily used in the stability theory for ordinary differential equations. Of fundamental importance in the latter theory is the fact that solutions of the non-homogeneous equation

$$\frac{dz}{dt} = A(t)z + w(t) \tag{5}$$

can be represented by a simple integral operator involving w. In fact, if $Y(t)$ denotes the matrix solution of

$$\frac{dY}{dt} = A(t)Y, \qquad Y(0) = I, \tag{6}$$

where I is the identity matrix, then

$$z(t) = \int_0^t Y(t)Y^{-1}(s)w(s)\,ds \tag{7}$$

is the particular solution of equation (5) for $t > 0$ which satisfies the condition $z(0) = 0$. The usual derivation of this result utilizes the method of variation of parameters,[2] and depends on an independent proof that $Y^{-1}(t)$ exists for $t \geq 0$. Such a proof is not difficult, for differential equations.[3] However, the method fails when applied to more complicated functional equations such as the non-homogeneous counterpart of the equation in (3).

It turns out that a more illuminating approach is furnished by use of the concept of the *adjoint equation*. Let us illustrate this first for the differential system in (5). If we multiply this system by a matrix Y, as yet unspecified, and integrate, we obtain the relation

$$\int_0^t Y(s)z'(s)\,ds = \int_0^t Y(s)A(s)z(s)\,ds + \int_0^t Y(s)w(s)\,ds, \qquad t \geq 0.$$

After an integration by parts, this takes the form

$$Y(t)z(t) = \int_0^t \{Y'(s) + Y(s)A(s)\}z(s)\,ds + \int_0^t Y(s)w(s)\,ds, \quad t \geq 0, \tag{8}$$

if we assume that $z(0) = 0$. In order to simplify this equation, we now ask that Y satisfy the equation

$$Y'(s) + Y(s)A(s) = 0, \qquad 0 \leq s < t. \tag{9}$$

In order to avoid the use of the inverse matrix $Y(t)^{-1}$, let us impose the further condition

$$Y(t) = I. \tag{10}$$

[2] Cf. [1], page 11, for this derivation.

[3] [1], page 10.

Provided that $A(t)$ is integrable, the equation in (9) possesses a unique solution $Y(s)$ satisfying (10) and defined for $t \geq s \geq 0$. With this choice of Y, we obtain from the relation in (8)

$$z(t) = \int_0^t Y(s)w(s)\,ds. \tag{11}$$

This equation provides the desired integral representation for $z(t)$.

The systems in (9) and (6) are said to be *adjoint* to one another. The function Y actually depends on two variables, s and t, and the relations in (9), (10), and (11) can more explicitly be written in the forms

$$\frac{\partial}{\partial s} Y(s, t) = -Y(s, t)A(s), \qquad t > 0, 0 \leq s \leq t, \tag{12}$$

$$Y(t, t) = I, \tag{13}$$

and

$$z(t) = \int_0^t Y(s, t)w(s)\,ds, \tag{14}$$

respectively.

It is easy to verify that if $X(t)$ is the unique solution of the relations in (6), then the function $Y(s, t) = X(t)X^{-1}(s)$ is the unique solution of the relations in (12) and (13). Therefore (14) and (7) are equivalent results.

For differential-difference equations, the use of the inverse is no longer possible, but the adjoint method is applicable. We have shown in [3], for example, that the unique continuous solution of

$$z'(t + h_m) + \sum_{k=0}^{m} A_k(t)z(t + h_k) = w(t), \qquad t > 0, \tag{15}$$

$$z(t) = 0, \qquad 0 \leq t \leq h_m, \tag{16}$$

is given by the formula

$$z(t + h_m) = \int_0^t Y(s, t)w(s)\,ds, \qquad t > 0, \tag{17}$$

where $Y(s, t)$ is the continuous solution of the adjoint system

$$-\frac{\partial}{\partial s} Y(s, t) + \sum_{k=0}^{m} Y(s + h_m - h_k, t)A_k(s + h_m - h_k) = 0, \tag{18}$$
$$t > 0, 0 < s < t - h_m \text{ and } t - h_m < s < t,$$

with condition

$$Y(s, t) = \begin{cases} 0, & t < s \leq t + h_m, \\ I, & s = t. \end{cases} \tag{19}$$

From this representation, it is rather easy to establish the following stability theorem.

THEOREM. *Let $A_k(t)$ and $B_k(t)$ be continuous for $t > 0 (k = 0, 1, \cdots, m)$. Then a sufficient condition for all continuous solutions of Equation* (4) *to be bounded as $t \to +\infty$ is that*

(i) *all continuous solutions of Equation* (3) *be bounded;*

(ii) $\int^\infty \| B_k(t) \| \, dt < \infty, \quad k = 0, 1, \cdots, m;$

(iii) $\| Y(s, t) \| \leq c_1, \quad t \geq 0, 0 \leq s \leq t,$

where c_1 is a constant and $Y(s, t)$ is the adjoint matrix.

Complete details and additional results can be found in the paper of the authors, [3]. In particular, we obtain simplifications when the $A_k(t)$ are constants, and discuss a somewhat broader class of equations than is indicated in (3) and (4).

2. Asymptotic expansions

We have also considered the problem of determining the asymptotic behavior of solutions of linear differential-difference equations, the coefficients in which possess asymptotic power series expansions. The corresponding problem for a system of ordinary differential equations is to consider a system of the form in (1), where

$$A(t) \sim \sum_{k=0}^{\infty} A_k t^{-k}. \tag{20}$$

This problem has been extensively investigated (refer to Bellman, [1], for further discussion and references). For example, it is known that if the matrix A_0 has simple characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_n$, then with each root λ_j there is associated a solution $x_j(t)$ having an asymptotic expansion of the form

$$x_j(t) \sim e^{\lambda_j t} t^{r_j} \sum_{k=0}^{\infty} c_k t^{-k} \qquad (c_0 \neq 0), \tag{21}$$

where r_j is dependent on A_1 and where the c_k are constant vectors. Furthermore, since these n solutions are linearly independent, every solution is a linear combination of these particular solutions. If the characteriestic roots are not all simple, similar, but more complicated results are known. Also, less precise results have been found[4] if the relation in (20) is replaced by the weaker hypothesis

$$A(t) = A_0 + A_1(t) + A_2(t) \tag{22}$$

where

$$\int^\infty \| A_1'(t) \| \, dt < \infty, \qquad \int^\infty \| A_2(t) \| \, dt < \infty.$$

The proof of these results rests on diagonalization of the matrix $A_0 + A_1(t)$.

[4] Cf. [1], [5], [7].

The problem for differential-difference equations[5] is considerably more complicated, largely because the infinite-dimensional nature of such equations prevents use of a device as simple as diagonalization. Nevertheless, by means of a sequence of transformations we are able to reduce the problem to a form where the standard differential equation theory can be employed. We first transform the differential-difference equation into an integral equation, then transform this integral equation into an integro-differential equation. At this point the Liouville transformation plays a vital role. Although the guiding ideas are simple, the analysis becomes formidable, and we shall accordingly sketch the details here only in the simplest possible case. For a detailed discussion of this and other cases, refer to [4].

Consider the scalar differential-difference equation of first order

$$u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - h) = 0, \tag{23}$$

where a_0, b_0, h are given constants, $h > 0$, and where

$$a(t) \to 0 \quad \text{and} \quad b(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{24}$$

Let λ denote a root of the characteristic equation

$$s + a_0 + b_0 e^{-hs} = 0, \tag{25}$$

and assume that λ is real and simple, and that every other root has real part no greater than the real part of λ. The first step in our discussion is to convert the equation in (23) into an integral equation of the form[6]

$$u(t) = ce^{\lambda t} - \int_{t_0}^{t} a(r)u(r)k(t - r)\,dr. \tag{26}$$

Here $k(t)$ represents a certain kernel function of the form

$$k(t) = \sum e^{\lambda_n t} q_n(t), \tag{27}$$

the sum being taken over all roots λ_n of the equation in (25), and the $q_n(t)$ being polynomials. The equations in (26) and (27) can be obtained by transform theory, and are well-known.[7] Since there are constants c_1 and k such that

$$k(t) = c_1 e^{\lambda t} + k_1(t), \qquad |k_1(t)| = 0(e^{kt}), \qquad k < \lambda, \tag{28}$$

under the assumptions on λ we have made, we can replace the equation in (26) by

$$u(t) = ce^{\lambda t} - c_1 e^{\lambda t} \int_{t_0}^{t} e^{-\lambda r} a(r)u(r)\,dr + p(t), \tag{29}$$

$$p(t) = -\int_{t_0}^{t} a(r)u(r)k_1\,(t - r)\,dr. \tag{30}$$

[5] Previous work on this problem can be found in [6] and [8].

[6] For purposes of exposition we have taken $b(t) \equiv 0$, but this is not essential.

[7] Bellman, [2].

If we make the assumption that $\int^{\infty} |a(t)| \, dt < \infty$, we can use the equation in (29) directly to show that $u(t)$ must be asymptotic to a constant multiple of $e^{\lambda t}$. Since, however, this is too severe a restriction, we must obtain a more suitable integral equation. We accordingly differentiate to obtain the following integro-differential equation.

$$\frac{d}{dt}\{u(t) - p(t)\} = \{\lambda - c_1 a(t)\}\{u(t) - p(t)\} - c_1 a(t) p(t). \tag{31}$$

The form of this equation suggests the "Liouville" transformation

$$s(t) = \int_{t_0}^{t} \{\lambda - c_1 a(r)\} \, dr, \tag{32}$$

which results in the equation

$$\frac{d}{ds}\{u(t) - p(t)\} = u(t) - p(t) - \frac{c_1 a(t) p(t)}{\lambda - c_1 a(t)}.$$

It follows that there is a solution $u(t)$ of Equation (23) which satisfies the improved integral equation

$$u(t) = e^{s(t)} + p(t) - c_1 e^{s(t)} \int_{t_0}^{t} e^{-s(r)} a(r) p(r) \, dr, \qquad t \geq t_0. \tag{33}$$

From this equation it can readily be shown that

$$u(t) = e^{s(t)}\{c_2 + 0(1)\}, \qquad t \to \infty, \tag{34}$$

provided $a(t)$ satisfies conditions such as

$$\int^{\infty} \{a(t)\}^2 \, dt < \infty, \qquad \int^{\infty} |a'(t)| \, dt < \infty. \tag{35}$$

Furthermore, if $a(t)$ possesses an asymptotic power series expansion, then so does $u(t)e^{-s(t)}$.

By more complicated analysis, we can handle cases in which λ is complex or multiple, or in which it is not the root of greatest real part. Thus we can conclude that with each root λ for the equation in (25) there is associated a solution with a certain asymptotic behavior which can be determined from the coefficients in (23) by definite procedures, provided $a(t)$ and $b(t)$ satisfy conditions similar to those in (24) and (35). Our methods also apply to higher order differential-difference equations, and indeed to any linear functional equations for which representations such as (26) and (27) are available.

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