FORCED OSCILLATIONS IN 3-SPACE*

By Courtney Coleman

1. Introduction

Let x be an n-vector, λ a real parameter, and $f(x, t)$ and $g(x, t, \lambda)$ n-vector functions of period 1 in t. Topological methods have been used quite frequently to show that the system,

$$dx/dt = f(x, t) + g(x, t, \lambda), \tag{1.1}$$

has a solution of period 1, a forced oscillation, if f, g, λ, and n satisfy certain conditions. Bass ([1]), Berstein ([2]), and Halanay ([6]) have used the notion of topological degree for small λ if (1.1) is known to have a periodic solution for $\lambda = 0$. Lefschetz ([8]), Levinson ([9]), and others have used the Brouwer theorem for $\lambda = 0$, $n = 2$. For this case Cronin ([4]), reinterpreting and extending the results of Gomory ([5]), has used, not the Brouwer theorem, but more general theorems involving topological degree.

In this paper, we consider the case $\lambda = 0$, $n = 3$, and we state that under certain conditions (1) has a periodic solution of period 1. The basic idea is an extension to 3-space of Cronin's methods in [4]. Section 2 contains the main theorems and some explanatory remarks. No proofs are given here. The last section is devoted to some examples.

For the specific ideas used in this paper the author is indebted to the papers mentioned above by Cronin and by Gomory. For the topological approach to the study of differential systems and for the general ideas of the behavior of trajectories near singularities the author owes a great deal to Professor Lefschetz and to his book ([8]).

2. Statement of theorems

Consider the system,

$$dx/dt = f(x) + e(x, t), \tag{2.1}$$

where x is an n-vector, $f(x)$ and $e(x, t)$ have continuous first derivatives and e has least period 1 in t. If $x = x(t)$ is a solution of (2.1), it may happen that

$$\| x(t) \|,$$

the Euclidean norm, tends to infinity as t tends to some finite limit, t_0. In the next paragraphs in order to define certain transformations no trajectory may have such a finite "escape time". To insure this, we proceed as follows. Let A_0 be

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an n-cell. Then it is possible to find a positive scalar function $\mathfrak{A}(x)$ with the properties: a) $\mathfrak{A}(x) \equiv 1$ on A_0; b) $\mathfrak{A}(x)$ tends to 0 so rapidly as $\| x \|$ tends to infinity that $(f(x) + e(x, t))\mathfrak{A}(x)$ is bounded for all x, t; c) $\mathfrak{A}(x)$ has continuous first derivatives. Thus no trajectory of the system,

$$(2.2) \qquad dx/dt = (f(x) + e(x, t))\mathfrak{A}(x)$$

can have a finite "escape time" since the right hand side is bounded everywhere. The integral curves of (2.2) coincide with those of (2.1) in phase space, and within A_0 the parametrizations also coincide. Hence, any solution of (2.2) lying in A_0 for all t is also a solution of (2.1), and conversely.

Let $x = x(t, t^0, x^0)$ be the solution of (2.2) for which $x(t^0, t^0, x^0) = x^0$. Let T_t be the mapping, $x^0 \rightarrow x(t, 0, x^0) - x^0$ for $t > 0$. T_t is continuous for all $t > 0$ and for all x^0. Let T' be the mapping $x^0 \rightarrow f(x^0) + e(x^0, 0)$. T' is continuous for all x^0. An n-cell A is said to have *property* I if A contains the origin and all zeros of $f(x) + e(x, 0)$ and if $x(t, t^0, x^0) \neq x^0$ for any $t > t^0$, $x^0 \in \dot{A}$, the boundary of A. The topological degrees of T_t and of T' are defined at the origin relative to an n-cell A with property I.

THEOREM 1. *Let A be an n-cell with property* I. *Then for any $t > 0$ the topological degrees of T_t and T' at the origin relative to A are equal.*

For $n = 2$, this theorem is proved by Cronin in [4]. For $n \geqq 3$, the proof is much the same.

THEOREM 2. *Let A be an n-cell with property* I. *If the topological degree of T' at the origin and relative to A is not zero, then* (2.2) *has a periodic solution of period* 1 *lying in A for all t.*

The conclusion follows directly from Theorem 1 and a standard fixed point theorem (Theorem 2 in [7], for example).

In certain cases of interest in this paper the computation of the topological degree of T' can be simplified. Let $f(x)$ and $e(x, t)$ be vector polynomials in $x(e(x, t)$ with coefficients which are periodic functions of $t)$ of integral degrees k and m respectively, $k > m \geqq 0$. Let $f^{(k)}(x) = (f_1^{(k)}(x), \cdots, f_n^{(k)}(x))$ denote the vector whose ith component is the homogeneous polynomial containing the terms of degree k of $f_i(x)$. Suppose that the functions $f_i^{(k)}(x)$ have no common zero. Let T'' be the mapping, $x^0 \rightarrow f^{(k)}(x^0)$. The topological degree of T'' at the origin and relative to any n-cell A containing the origin is defined. If A is an n-cell with property I, then A is said to have *property* II d, for $d > 0$, if the distance from the origin to $\dot{A}$ is greater than d. Theorem 3 can then be proved without difficulty.

THEOREM 3. *Let $f(x)$ and $e(x, t)$ satisfy all the conditions given in the preceding paragraph. For any $d > 0$ let there be at least one n-cell, $A(d)$, having properties* I *and* IId. *Then there is a $d_0 > 0$ such that for $d > d_0$ the topological degrees of T' and T'' at the origin and relative to $A(d)$ are equal.*

Note that if the n-cell $A(d)$ is contained within the n-cell A_0, any solution of (2.2) lying in $A(d)$ for all t is also a solution of (2.1). The question of the existence of a forced oscillation for (2.1) thus reduces to: (a) giving conditions sufficient to ensure the existence of an n-cell $A(d) \subset A_0$ with properties I and IId; and (b) giving conditions that the topological degree of T'' at the origin relative to $A(d)$ be non-zero. Theorem 4 relates to problem (a) for $n = 3$, while about (b) we note only that the degree is non-zero for odd k.

Let $f(x) = \sum_{j=0}^{k} f^{(j)}(x)$, $e(x, t) = \sum_{j=0}^{m} e^{(j)}(x, t)$, where $f^{(j)}(x)$ and $e^{(j)}(x, t)$ are vectors whose components are homogeneous polynomials in x of degree j, and $k > m \geqq 0$. If $u = \| x \|$, $y = ux$, (2.1) becomes

$$(2.3) \qquad \begin{aligned} &\text{a.} \quad du/ d\tau = -u(y, f^{(k)}(y)) + u^2(y, P(u, y, t)) \\ &\text{b.} \quad dy/ d\tau = g(y) + uQ(u, y, t), \end{aligned}$$

where $d\tau = u^{1-k}\, dt$, (,) denotes scalar product,

$$P(u, y, t) = -\sum_{j=0}^{k-1} f^{(j)}(y)u^{k-j-1} - \sum_{j=0}^{m} e^{(j)}(y, t)u^{k-j-1},$$

$$g(y) = f^{(k)}(y) - (y, f^{(k)}(y))y,$$

$$Q(u, y, t) = -(y, P(u, y, t))y - P(u, y, t).$$

Let S be the integral surface of (2.3) defined by $u = 0$. The trajectories of (2.3) on S satisfy the system,

$$(2.4) \qquad dy/ d\tau = g(y).$$

A trajectory γ of (2.4) on S is called a critical trajectory if it is a critical point y^0 of (2.4) or if it is a cycle c, $y = y(\tau)$, of period ω_c. Let $h(\gamma)$, for a critical trajectory γ, be $(y^0, f^{(k)}(y^0))$ in the first case and $\int_0^{\omega_c} (y(\tau), f^{(k)}(y(\tau)))\, d\tau$ in the second. If γ is a critical trajectory, it is called attractive if $h(\gamma) > 0$, repulsive if $h(\gamma) < 0$.

Let y^0 be a critical point of (2.4). It is said to be an elementary critical point relative to S if the determinant of the coefficients of the linear terms of g expressed in suitable surface coordinates centered on y^0 is not zero. That is, let $y^0 = (y_1^0, \cdots, y_n^0)$ and suppose, without loss of generality, that $y_n^0 > 0$. In a suitable neighborhood of y^0, $y_n = (1 - y_1^2 - \cdots - y_{n-1}^2)^{1/2}$. Let

$$z = (z_1 \cdots, z_{n-1}),$$

where $z_i = y_i - y_i^0$. Then (2.4) becomes

$$(2.5) \qquad dz/d\tau = G_0 \cdot z + 0(\| z \|^2),$$

where G_0 is a constant $n - 1 \times n - 1$ matrix. The critical point y^0 is elementary if $\det G_0 \neq 0$.

THEOREM 4. *Hypotheses*: (a) $f(x) = \sum_{j=0}^{k} f^{(j)}(x)$ *and*

$$e(x, t) = \sum_{j=0}^{m} e^{(j)}(x, t),$$

where $f^{(j)}(x)$ *and* $e^{(j)}(x, t)$ *are vectors whose components are homogeneous polynomials in* x *of degree* j *and where* $k > m \geqq 0$;

(b) $n = 3$;

(c) *the critical trajectories of* (2.4) *are finite in number, for no critical trajectory* γ *is* $h(\gamma) = 0$, *and the critical points on* S *are elementary relative to* S;

(d) *there are no closed graphs of trajectories on* S;

(e) *If the alpha limit set,* $A(\gamma)$, *of a trajectory* γ *on* S *contains an attractive critical trajectory, then its omega limit set,* $\Omega(\gamma)$, *does not contain a repulsive critical trajectory.*

Conclusion: *There is a* $d_0 > 0$ *such that for each* $d > d_0$ *there are a* 3*-cell* A_0 *and its associated scalar function* $\mathfrak{A}(x)$ (*as defined previously*) *and a* 3*-cell* $A(d) \subset A_0$ *such that* $A(d)$ *has properties* I *and* IId (A_0 *may depend on* d).

Remark 1. The hypothesis that $h(\gamma) \neq 0$ if γ is a critical point y^0 of (2.4) implies that $y^0 = (y_1^0, \cdots, y_n^0)$ is a solution of the system

$$f_1^{(k)}(y)/y_1 = \cdots = f_n^{(k)}(y)/y_n .$$

Conversely, any solution of this system is a critical point. This same hypothesis also implies, evidently, that the $f_i^{(k)}(x)$ have no zero in common—a hypothesis for Theorem 3.

Remark 2. Hypothesis (a) is needed in order that (2.1) might be transformed into (2.3). The particular form of (2.3) is essential in the construction of $A(d)$. (b) is needed in order that S be the 2-sphere for which the Poincaré-Bendixson theory is applicable. As for (c), the mechanics of the proof require that for a critical trajectory γ, $h(\gamma) \neq 0$. Actually, the other two clauses in (c) could be dropped; but, for simplification, we leave these clauses in. (d) is used in constructing canonical sets on S and (e) is essential in the actual construction of $A(d)$.

Remark 3. Hypotheses (a), (c), (e) are extensions of Gomory's hypotheses for a system in 2-space.

Remark 4. If the hypotheses of Theorem 4 are satisfied and if k is odd, then (2.2) has at least one forced oscillation contained within the n-cell A_0 (actually within $A(d)$). As noted before, such a solution of (2.2) is also a solution of (2.1).

Remark 5. Nothing will be said here about the number of forced oscillations. The reader is referred to [4] for a discussion of this question for $n = 2$.

The proof of Theorem 4 is very long and will be given elsewhere.

3. Examples

We first note some general properties of the system,

$$dx/dt = f^{(k)}(x), \tag{3.1}$$

where x is a 3-vector and $f^{(k)}(x)$ is a vector whose components are homogeneous polynomials of degree $k \geqq 1$ in x. If $y = x \| x \|^{-1}$ then we have,

$$dy/d\tau = f^{(k)}(y) - (y, f^{(k)}(y))y, \tag{3.2}$$

where $d\tau = dt \| x \|^{k-1}$. (3.2) gives a vector field on the unit two sphere, S^2, centered at the origin in the x-space. The following statements are proved in [3].

(A) If γ is an integral curve of (3.1), then its projection onto S^2 via rays through the origin is an integral curve of (3.2).

(B) If γ is an integral curve of (3.2) on S^2, then the cone generated by the rays through the origin and the points of γ is an integral surface of (3.1).

(C) Every integral curve of (3.1) has a positive limiting direction and a negative limiting direction if and only if (3.2) has no closed graphs and no cycles.

(D) Let attractive and repulsive critical trajectories on S^2 be defined as in section 2. Then (3.1) has a trajectory whose alpha and omega limit sets contain the origin if and only if there is a trajectory of (3.2) whose alpha limit set contains an attractive critical trajectory while its omega limit set contains a repulsive critical trajectory.

The verification of hypotheses (c), (d), (e) of Theorem 4 can often be simplified by using these properties.

Example 1. Consider the linear system

$$dx/dt = Ax + e(t), \tag{3.3}$$

where A is a constant 3×3 matrix and $e(t)$ is continuously differentiable and has least period 1 in t.

First suppose that $Ax = (r_1x_1, r_2x_2, r_3x_3)$, where the r_i are real, distinct, and non-zero. Then there are six critical points on S, $\gamma_{11} = (1, 0, 0), \cdots, \gamma_{32} = (0, 0, -1)$, and $h(\gamma_{ij}) = r_i$, $i = 1, 2, 3$, $j = 1, 2$. There are no cycles on S. It is not hard to show that hypotheses (d) and (e) of Theorem 4 are satisfied in this case.

Next suppose that $Ax = (r_1x_1 + r_2x_2, -r_2x_1 + r_1x_2, r_3x_3)$, where the r_i are real, non-zero, and $r_1 \neq r_3$. On S there are two critical points $(0, 0, \pm 1)$, and for each $h = r_3$.

There is one cycle on S, $y_3 = 0$, and for it $h = 2\pi r_1 | r_2 |^{-1}$. Again, it is easy to verify hypotheses (d) and (e) of Theorem 4. Since $k = 1$, (3.3) has a forced oscillation in each of these cases. We do not examine here the other possibilities for Ax.

Example 2. Consider the system

$$dx_i/dt = r_ix_i^k + e_i(x, t), \qquad i = 1, 2, 3, \tag{3.4}$$

where the r_i are real and non-zero, $k = 2s + 1$ for some positive integer s, and the $e_i(x, t)$ are polynomials in x of degree no greater than $k - 1$ with continuously differentiable coefficients which have period 1 in t. If the r_i have a common sign, it can be shown that there are on S twenty-six critical points, all elementary, and no cycles. If the r_i do not all have a common sign, it can be shown that on S there are no cycles and ten critical points, again elementary. In either case, the hypotheses of Theorem 4 can be verified. Since k is odd, it follows that (3.4) has a forced oscillation.

RIAS, BALTIMORE, MARYLAND AND HARVEY MUDD COLLEGE, CLAREMONT, CALIFORNIA

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