ON BOUNDED SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS

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1. Introduction

We shall consider a system of equations in the following form:

$$dx/dt = f(t, x, \mu), \tag{1.1}$$

where t, x, and μ are a real variable, an n-dimensional vector and an m-dimensional vector, respectively; components of x and μ are complex. The vector f satisfies condition

$$f(t, 0, 0) = 0, \tag{1.2}$$

and it is given in a form of expansion in powers of components of x and μ:

$$f(t, x, \mu) = Ax + B(t)\mu + \sum_{|p|+|q|\geqq 2} f_{pq}(t)x^p\mu^q, \tag{1.3}$$

where p and q are systems of nonnegative integers $p_1, \cdots, p_n$, and $q_1, \cdots, q_m$, respectively; A and $B(t)$ are an n by n matrix and an n by m matrix respectively; the f_{pq} are n-dimensional vectors; and

$$x^p = x_1^{p_1} \cdots x_n^{p_n},$$

$$\mu^q = \mu_n^{q_1} \cdots \mu_m^{q_m},$$

$$p = p_1 + \cdots + p_n,$$

$$q = q_1 + \cdots + q_m,$$

the x_j and μ_j being components of x and μ. We shall suppose that A is a constant matrix, and that components of $B(t)$ and $f_{pq}(t)$ are almost periodic in t. The series representing the vector f is assumed to converge uniformly for

$$\| x \| \equiv \max_j | x_j | \leqq \delta_0, \qquad \| \mu \| \equiv \max_j | \mu_j | \leqq \rho_0, \qquad -\infty < t < +\infty, \tag{1.4}$$

where δ_0 and ρ_0 are positive numbers.

Let

$$x = \phi(t, C, \mu) \tag{1.5}$$

be a family of solutions of the system (1.1) depending on a vector C of arbitrary constants $C_1, \cdots, C_r$. The constants C_j are assumed to be complex.

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THEOREM. *Suppose that the family* (1.5) *satisfies the following conditions*:

(i) ϕ *is continuous and bounded in* (t, C, μ), *and holomorphic in* (C, μ) *for*

$$\| C \| \leqq \delta, \qquad \| \mu \| \leqq \rho, \qquad 0 \leqq t < +\infty, \tag{1.6}$$

where δ *and* ρ *are positive numbers*:

(ii) ϕ *is identically equal to zero for* $C = 0$, $\mu = 0$, $0 \leqq t < +\infty$, *i.e.*

$$\phi(t, 0, 0) \equiv 0. \tag{1.7}$$

Then ϕ *approaches asymptotically a family of almost periodic solutions of the system* (1.1) *as* t *goes to* $+\infty$.

The proof of our theorem is based on the following lemma:

Let A be a constant matrix, and $h(t)$ and $k(t)$ be n-dimensional vectors which satisfy the conditions: (i) $k(t)$ is almost periodic in t; (ii) $\| h(t) \| \leqq Ke^{-\sigma t}$ for $0 \leqq t < +\infty$, where K and σ are positive numbers. Consider a system of linear equations:

$$dy/dt = Ay + h(t) + k(t). \tag{1.8}$$

LEMMA. *Every solution of* (1.8) *which is bounded for* $0 \leqq t < +\infty$ *can be given the form*

$$y(t) = a(t) + b(t), \tag{1.9}$$

where $a(t)$ *and* $da(t)/dt$ *are almost periodic in* t, *and*

$$\begin{aligned} \| b(t) \| &\leqq K' e^{-\sigma' t}, \\ \| db(t)/dt \| &\leqq K' e^{-\sigma' t} \end{aligned} \tag{1.10}$$

for $0 \leqq t < +\infty$, K' *and* σ' *being positive numbers.*

As an example, let us consider a system of the following form:

$$dx/dt = Ax + f(t, x) + \mu g(t, x, \mu), \tag{1.11}$$

where the right-hand member satisfies conditions similar to those satisfied by that of (1.1) except that μ is a single parameter. Suppose that the real parts of characteristic values of the matrix A are all distinct from zero. Let r be the number of those characteristic values whose real parts are negative. Then we can prove that the system (1.11) has a family of solutions satisfying the conditions given in our theorem. For this case, the existence of almost periodic solutions was proved in several papers (see Bibliography).

2. Proof of theorem

Let

$$\phi(t, C, \mu) = \sum_{|p|+|q| \geqq 1} \phi_{pq}(t) C^p \mu^q. \tag{2.1}$$

Since ϕ is bounded for $\| C \| \leqq \delta$, $\| \mu \| \leqq \rho$, $0 \leqq t < +\infty$, there exists a positive number M such that

$$\| \phi_{pq}(t) \| \leqq M\delta^{-|p|}\rho^{-|q|} \tag{2.2}$$

for $0 \leqq t < +\infty$. On the other hand, since ϕ is a solution of (1.1), we can prove, by the use of lemma, that $\phi_{pq}(t)$ and $d\phi_{pq}(t)/dt$ can be given the forms

$$\begin{aligned} \phi_{pq}(t) &= a_{pq}(t) + b_{pq}(t), \\ \phi'_{pq}(t) &= a'_{pq}(t) + b'_{pq}(t), \end{aligned} \tag{2.3}$$

where $'$ stands for d/dt, the a_{pq}, a'_{pq} are almost periodic in t, and the b_{pq}, b'_{pq} tend to zero as t goes to $+\infty$. Furthermore, there exists another positive constant M' such that

$$\| \phi'_{dq}(t) \| \leqq M'\delta^{-|p|}\rho^{-|q|} \tag{2.4}$$

for $0 \leqq t < +\infty$, because ϕ is a solution of (1.1). Therefore, we can get estimates:

$$\| a_{pq}(t) \| \leqq M\delta^{-|p|}\rho^{-|q|}, \quad \| a'_{pq}(t) \| \leqq M'\delta^{-|p|}\rho^{-|q|} \tag{2.5}$$

for $-\infty < t < +\infty$, and

$$\| b_{pq}(t) \| \leqq 2M\delta^{-|p|}\rho^{-|q|}, \quad \| b'_{pq}(t) \| \leqq 2M'\delta^{-|p|}\rho^{-|q|} \tag{2.6}$$

for $0 \leqq t < +\infty$. Hence ϕ can be given the form

$$\phi(t, C, \mu) = a(t, C, \mu) + b(t, C, \mu) \tag{2.7}$$

and

$$\phi'(t, C, \mu) = a'(t, C, \mu) + b'(t, C, \mu), \tag{2.8}$$

where

$$a(t, C, \mu) = \sum_{|p|+|q| \geqq 1} a_{pq}(t) C^p \mu^q. \tag{2.9}$$

It is readily seen that a and a' are almost periodic in t, and b and b' tend to zero as t goes to $+\infty$. Finally

$$a' - f(t, a, \mu) = f(t, a + b, \mu) - f(t, a, \mu) - b' \tag{2.10}$$

implies that a is a solution of (1.1). In fact, the right-hand member of (2.10) is almost periodic in t, whereas the left-hand member tends to zero as t goes to $+\infty$. Therefore $a' - f(t, a, \mu)$ must be identical to zero. The theorem is thus established.

Remark. A result similar to our theorem was proved in one of our previous papers [6] in the case when the right-hand member of (1.1) does not contain any parameter. Our theorem is an immediate consequence of this previous result, because by adding to (1.1) equations $d\mu/dt = 0$, the present case is reduced to the previous one. However, by the use of our lemma, we could simplify our previous proof.

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