TURNING-POINT PROBLEMS IN DIFFRACTION THEORY

(Summary)

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Consider the boundary value problem

$$[\nabla^2 + \omega^2]v = \delta(x) \quad \text{for} \quad x \in \text{ext } B \subset E^3,$$

$$\partial v/\partial n|_{\partial B} = 0, \quad \text{and} \quad v \in L_2(\text{ext } B),$$

where B is a prolate spheroid or elliptic cylinder and $\delta(x)$ is a source distribution with compact support lying in ext B. The solution v of this problem is often thought of as being the steady-state solution of the mixed problem

$$\nabla^2 u - u_{tt} = \delta(x)e^{i\omega t} \quad (t > 0)$$

$$\partial u/\partial n|_{\partial B} = 0, \; u(x, 0^+) = f(x), \text{ and } u_t(x, 0^+) = g(x).$$

The method which R. K. Ritt and I have used to determine $v(x)$ is as follows. The initial key is given by an Abelian Theorem of Ritt which substantiates a trick which physicists have used for some time.

Theorem. *If $v(x, t) = u(x, t)e^{-i\omega t}$ and f, g, and δ have compact support, then the Laplace transform $V(x, s)$ of $v(x, t)$ is in $L^2(\text{ext } B)$ and $w(x, s) = sV(x, s)$ is such that*

$$(*) \qquad \lim_{s \to 0^+} w(x, s) = v(x),$$

$$(**) \qquad [\nabla^2 + (\omega - is)^2]w = \delta,$$

and

$$w \in L^2(\text{ext } B).$$

Of course, w also satisfies the boundary condition at B. It turns out to be easier to determine $w(x, s)$ and then to find $v(x)$ via $(**)$ rather than to derive the form of v directly.

Equation $(*)$ can be written in the form $-(L_\xi + L_\eta)w = J(\xi, \eta)\delta$ where ∂B is defined by $\xi = \xi_0$ and where the operators L_ξ and L_η are dependent upon ξ and η alone, respectively.

The operator $L_\xi - \mu$ is not self-adjoint so that classical Sturm-Liouville theory cannot be used to find a representation for w. However, Sims and Phillips have provided a resolvent theory for operators of the same type as $L_\xi - \mu$ and $L_\eta - \nu$. Their work guarantees that there exist resolvents R_μ and $\tilde{R}_\nu$ for $L_\xi - \mu$ and $L_\eta - \nu$, respectively, with the property that for certain paths Γ in the

μ-plane,

$$\Gamma: l + ics$$

$(-\infty < l < \infty$, c a positive constant depending on ω and the dimensions of $B)$,

$$w(\xi, \eta, s) = \frac{1}{2\pi i} \int_\Gamma \tilde{R}_{-\mu} R_\mu[J\delta] \, d\mu.$$

A path Γ separates the poles of $\tilde{R}_\mu$ and R_μ. If δ reduces to a point source at (Ξ, τ) in the (ξ, η)-plane (a line source in cylinder problems), this integral simplifies and becomes

$$w(\xi, \eta, \Xi, \tau, s) = \frac{1}{2\pi i} \int_\Gamma \tilde{G}(\eta, \tau, -\mu) \, G(\xi, \Xi, \mu) \, d\mu,$$

where $\tilde{G}$ and G are the resolvent Green's functions for the operators L_ξ and L_η, respectively.

One may attempt to evaluate this integral as a residue series by considering either the poles of $\tilde{G}$ or those of G. The expansion involving the poles of the angular Green's function is a classical one, often called the Mie series expansion; but it converges so slowly when λ is small $(\lambda = 2\pi/\omega)$ as to be almost useless then. The expansion involving the poles of the radial Green's function corresponds to the use of the Watson transform, familiar in right circular cylinder and sphere problems. However, except in such simple cases the proper formulation of the Watson transform had not been found prior to this approach. In cases which have been studied thus far, the radial residue series is rapidly convergent, when λ is small, for points (ξ, η) in the geometric shadow of B.

It is in determining the terms of the radial residue series, particularly in locating the poles of G, that turning-point problems arise. Specific examples, including unsolved problems, were discussed. This work will appear in a joint paper with R. K. Ritt in the *Archive of Rational Mechanics and Analysis.*

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