NOTE ON A THREE-DIMENSIONAL SINGULARITY OF A CONTINUOUS VECTOR FIELD

By Samuel Barocio

In a note which appeared in "Contributions to the Theory of Nonlinear Oscillations," Vol. II, 1952, S. Lefschetz proved in a very elegant way that for two-dimensional singularities of analytical systems there are only five possible typical sectors, namely:

I. *Fans*, in which the trajectories all tend to the critical point or all away from it. The nodes are particular cases of fans.

II. *Hyperbolic Sectors*. The trajectories approach the singular point and then leave it.

III. *Nested Oval Sectors*. The trajectories issue from the origin and then reenter to it.

IV. *Foci*. The trajectories approach or leave the origin spiraling.

V. *Centers*. The trajectories in the neighborhood of the origin are all closed.

The outstanding feature of the above mentioned note is that the treatment rests chiefly on pure geometrical, i.e. topological, arguments. Now, extending these methods to the three dimensional case, one is tempted to think that here the *a priori* possibilities of the behaviour of the trajectories are the same. But unfortunately the situation in three dimensions is quite different; this is so because in two dimensions the basic tool is the Jordan curve theorem, but in three dimensions the possible geometrical structures are more complicated.

In the two dimensional cases the sectors are made up with families of curves which are homotopic among themselves, and in such way that during the homotopy they do not leave the family.

Now this condition cannot be taken for granted in three dimensions because, in this case, there arises the possibility of an infinite family of closed trajectories in every neighborhood of the origin which are knots. We are going to construct here an example of this sort in the case of a *continuous* field.

Let us consider the neighborhood of the origin in a rectangular system of coordinates x, y, z. Now, draw two lines AA' and BB' on the plane yz (fig. 1), making an angle α with the y-axis, and divide the triangular region $0AB$ with vertical lines aa', bb', cc', $\cdots$ in such way to have

$$\frac{0a}{0b} = \frac{0b}{0c} = \frac{0c}{0d} = \cdots = K,$$

i.e. an infinite family of similar triangles $0aa'$, $0bb'$, $\cdots$. Now revolve the zy plane around the z axis; then the trapezoidal area $aa'b'b$ describes a solid torus. The same happens with all the remaining trapezoids. Take now a braided knot, for example the well known one of figure 2. Of course it may be much more complicated, but what we want is that every loop be described in the same sense.

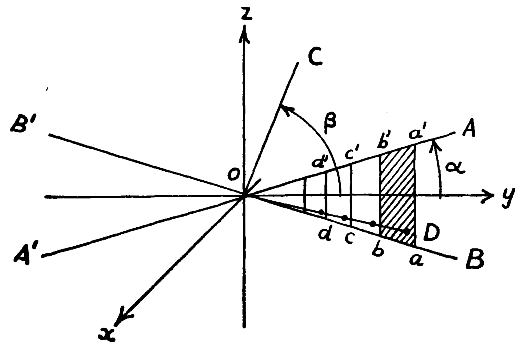

FIG. 1

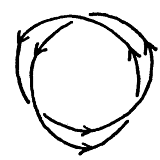

FIG. 2

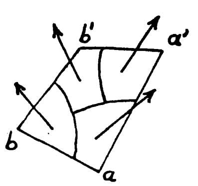

FIG. 3a

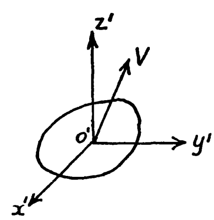

FIG. 3b

Now imbed the knot as trajectory in the torus described by $aa'b'b$. Then this trajectory crosses the trapezoid at as many points as the knot has loops, and all the crossings have the same sense. Now let us divide the trapezoid $aa'b'b$ into as many cells as there are crossings, making the intersection of the knot contained in the interior of the cell (fig. 3a). One easily realizes that these partitions can be extended continuously around all the solid torus by imagining the knot replaced by a thin tube. If one blows this tube till it fills the solid torus, then the intersections of the walls of the blown tube and the trapezoids are the boundaries of the cells. Take now a local frame of reference $x'y'z'$ on one cell such that $0'x'$ and $0'y'$ be in the plane of the cell and $0z'$ be normal to this plane, where $0'$ is the intersection with the knot and V the unit tangent vector to it (fig. 3b).

Now extend the vector field to fill the cell, projecting V on the three axes of the local system and varying along the line $0x'$ the z' projection linearly from its value at $0'$ to 1 on the boundary of the cell and the x' and y' projections from their values at $0'$ to 0 on the boundary. Make the same with all the remaining radii. Then we have extended the vector field to all the cell, the vector on the boundary of the cell being of unit length and normal to the cell plane. Hence we have filled each trapezoidal torus described by $a\ a'\ b'\ b$ with a continuous field which on the boundary of the torus is the fibration determined by its parallels.

Repeat now this construction by similarity for the infinite family of tori. Now we have filled the space between the two cones generated by AA' and BB'. On these two cones the field is normal to the generatrix in each point. Let B be the angle of $0y$ and $0C$ (fig. 1), and define the field on the cone generated by $0C$ as being tangent to the cone and making an angle with the generatrix $0C$ which varies linearly from $\pi/2$, when $\beta = \alpha$, to 0, when $\beta = \pi/2$. Hence there are two trajectories that enter to the origin along the z axis: on the cones generated by the $0C$ type lines the trajectories enter to the origin spiraling on the cone; on the cones generated by AA' and BB' the trajectories are circles. In each torus there is at least one knotted trajectory, and the unclosed trajectories are wandering. It is clear that this field can be "smoothed" in order that it be not only continuous but continuously differentiable.

An interesting modification of our example can be made by taking a segment like $0D$ that cuts every knot in one point (segments of this kind exist because of the similarity construction) and contracting it toward the origin continuously in such a way as to obtain an identification transformation, that is to say, a topological transformation in the complement of the $0D$ segment. In this case the knots are transformed into trajectories which issue from the origin; then they knot with themselves and finally reenter the origin.

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