LINEAR DIFFERENTIAL EQUATIONS AND FUNCTIONAL ANALYSIS

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Section 1

Generally speaking, the subject of this research is the study of certain properties of linear and quasi-linear differential equations by means of methods of functional analysis. The results obtained so far have been published in part; other results are in course of publication and there are several problems which are still under investigation.

The equations considered are

$$\dot{x} + A(t)x = 0 \tag{1}$$

$$\dot{x} + A(t)x = f(t) \tag{2}$$

$$\dot{x} + A(t)x = h(x, t) \tag{3}$$

where x is an element of a Banach space X; $A(t)$ an endomorphism of X and $f(t)$ an element of X, both being functions of the real independent variable $t \in J = [0, \infty)$, (Bochner) integrable in each finite subinterval; $\dot{x} = dx/dt$; h a function, generally non-linear, from $X \times J$ into X, which satisfies adequate regularity and boundedness assumptions.

A central problem in our work is the following: under what conditions is it true that (2) has at least one solution belonging to a given Banach function space $\mathbf{D}$ (a $\mathbf{D}$-solution) for each $f \in \mathbf{B}$, where $\mathbf{B}$ is another given Banach function space? If such is the case, we shall say that the pair $(\mathbf{B}, \mathbf{D})$ is admissible with respect to (2). The function spaces considered belong to the classes studied by J. J. Schäffer in [14], among which are all Orlicz spaces and a fortiori the spaces $\mathbf{L}^p$, $1 \leq p \leq \infty$; the main properties of these classes will be explained in Part 2 of this report. The special case $\mathbf{B} = \mathbf{D} = \mathbf{C}$, the space of bounded continuous functions in J, X being of finite dimension and A continuous, was studied by Perron [11], who may accordingly be considered as a pioneer in this field. Certain papers by Persidskiĭ [12], Malkin [5], Maĭzel' [4], Kreĭn [2] and Kučer [3] are also related to this subject; the last two use to a certain extent functional-analytic methods.

The incidence of functional analysis in our work arises from three different sources:

a) From the fact that X is a Banach space, the dimension of which is generally infinite. However this is neither the most important application of functional analysis nor the essential reason of the generality of the theorems proved; almost all theorems retain their significance in the case where X is a finite-dimensional Euclidean space.

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b) From the application of category theorems of functional analysis (even if $\dim X < \infty$).

c) From the properties of the classes of function spaces considered.

Similar remarks can be made concerning the substitution of the usual continuity conditions by assumptions of the type of Carathéodory's. The theorems would indeed retain all their value if they were stated under continuity hypotheses. Many proofs are however formally much simpler under the more general assumptions.

Section 2

In order to explain the essential features of the function spaces considered, we state the following definitions.

Given two linear topological spaces $\mathbf{F}$, $\mathbf{G}$, we shall say that $\mathbf{F}$ is *stronger than* $\mathbf{G}$ ($\mathbf{G}$ *weaker than* $\mathbf{F}$) if and only if $\mathbf{F}$ is algebraically a linear manifold in $\mathbf{G}$ an $\mathbf{F}$-convergence implies $\mathbf{G}$-convergence. It should be borne in mind that these relations are not strict, i.e. each space is stronger (weaker) than itself; strict relations are accordingly denoted by the negatives *not stronger* (*weaker*) *than.* If $\mathbf{F}$, $\mathbf{G}$ are Banach spaces, the usual category argument shows that the relation "$\mathbf{F}$ is stronger than $\mathbf{G}$" is equivalent to $\mathbf{F} \subset \mathbf{G}$.

The space $\mathbf{L}$ is the Fréchet space of all real-valued measurable functions (modulo null sets) on J with the topology of convergence in the mean on each finite subinterval.

A linear normed function space $\mathbf{F}$ of real-valued measurable functions on J shall be called a $\mathfrak{J}$-space if the following assumptions are satisfied:

a) $\mathbf{F}$ is stronger than $\mathbf{L}$;

b) if $\varphi \in \mathbf{F}$ and ψ is any measurable function such that $|\psi(t)| \leq |\varphi(t)|$ a.e., then $\psi \in \mathbf{F}$, $|\psi|_{\mathbf{F}} \leq |\varphi|_{\mathbf{F}}$ (where $|\ |_{\mathbf{F}}$ denotes the $\mathbf{F}$-norm);

c) $\mathbf{F} \neq \{0\}$;

d) if $\varphi \in \mathbf{F}$ and ψ is the function defined by $\psi(t) = \varphi(t - \tau)$ for $t \geq \tau$, $= 0$ for $0 \leq t \leq \tau$, where $\tau > 0$, then $\psi \in \mathbf{F}$ and $|\psi|_{\mathbf{F}} = |\varphi|_{\mathbf{F}}$ (ψ may be termed a right-translation of φ).

If the following additional assumption is satisfied:

d*) If $\varphi \in \mathbf{F}$ and ψ is the function defined by $\psi(t) = \varphi(t + \tau)$ for $t \geq 0$, where $\tau > 0$ (ψ is a left-translation of φ), then $\psi \in \mathbf{F}$,

we shall say that $\mathbf{F}$ belongs to the class $\mathfrak{J}^\star$ which is thus a subclass of $\mathfrak{J}$.

A space $\mathbf{F} \in \mathfrak{J}$ is said to be *locally closed* if and only if its unit sphere is closed in $\mathbf{L}$.

The definitions given so far extend as follows to spaces of functions with values in a Banach space X: the space $\mathbf{F}(X)$, where $\mathbf{F} \in \mathfrak{J}$ (or $\mathfrak{J}^\star$), consists of all functions $f: J \to X$ which are strongly measurable and such that the real-valued function $\| f \|$ (defined by $\| f \|(t) = \| f(t) \|$) belongs to $\mathbf{F}$; $\mathbf{F}(X)$ is normed by $|f|_{\mathbf{F}(X)} = |\, \| f \| \,|_{\mathbf{F}}$. With this definition $\mathbf{F} = \mathbf{F}(R)$, R being the space of real numbers. The argument X will be omitted when confusion is unlikely.

The relation "stronger than" induces a *partial ordering in the class* $\mathfrak{J}$. It turns out that the weakest space in $\mathfrak{J}(\mathfrak{J}^\star)$ is the locally closed Banach space $\mathbf{M}$ of all measurable functions φ such that $\sup_{t \in J} \int_t^{t+1} |\varphi(\tau)|\, d\tau < \infty$, with this supremum as norm. Among all locally closed $\mathfrak{J}^\star$-spaces there is a strongest one, namely the Banach space $\mathbf{T}$ consisting of all measurable functions φ such that $\sum_{n=0}^{\infty} \operatorname{ess\,sup}_{n \le t \le n+1} |\varphi(t)| < \infty$, with a suitable norm. Orlicz spaces (in particular, $\mathbf{L}^p$ spaces, $1 \le p \le \infty$) are a subclass of the locally closed $\mathfrak{J}^\star$-spaces; the set-theoretical intersection and the algebraic sum of $\mathbf{L}^1$ and $\mathbf{L}^\infty$, with suitable norms, are, respectively, the strongest and weakest of all Orlicz spaces.

In relation to our definition of an admissible pair, it is convenient to introduce a partial ordering of the $\mathfrak{J}$-pairs $(\mathbf{B}, \mathbf{D})$, i.e. pairs consisting of a $\mathfrak{J}$-space $\mathbf{B}$ and a $\mathfrak{J}^\star$-space $\mathbf{D}$: $(\mathbf{B}_1, \mathbf{D}_1)$ *is stronger than* $(\mathbf{B}_2, \mathbf{D}_2)$ if and only if $\mathbf{B}_1$ is weaker than $\mathbf{B}_2$ and $\mathbf{D}_1$ is stronger than $\mathbf{D}_2$. Since a stronger space contains fewer functions, it is clear that "$(\mathbf{B}_1, \mathbf{D}_1)$ is admissible with respect to (2)" is a stronger assumption on (2) than "$(\mathbf{B}_2, \mathbf{D}_2)$ is admissible".

Section 3

A selection of typical results from [9], which includes as very special cases several theorems of [6], is the following:

LEMMA 1. *If $f_n \to f$, $x_n \to x$ ($\mathbf{L}$-convergence) and if $\dot{x}_n + A(t)x_n = f_n(t)$ for each n, then x is (modulo null sets) a solution of (2) and the limit $x_n \to x$ is uniform on each finite interval.*

The Lemma says that the linear mapping $x \to \dot{x} + A(t)x$, defined for any indefinite integral $x(t)$ of a function $\dot{x} \in \mathbf{L}$, has a closed graph in the $\mathbf{L}$-topology.

THEOREM 1. *If $X_{0\mathbf{D}}$ denotes the linear manifold of the initial values of all $\mathbf{D}$-solutions of (1), $X_{0\mathbf{D}}$ is closed if and only if there exists a positive number S such that, for every $\mathbf{D}$-solution, $|x|_{\mathbf{D}} \le S \,\| x(0) \|$.*

THEOREM 2. *If $(\mathbf{B}, \mathbf{D})$ is an admissible pair of Banach function spaces, there is a positive number K such that, for each $\epsilon > 0$ and $f \in \mathbf{B}$, there exists a $\mathbf{D}$-solution $x(t)$ of (2) such that $|x|_{\mathbf{D}} \le (K + \epsilon)\,|f|_{\mathbf{B}}$.*

The proofs of Theorems 1 and 2 are based on category arguments.

In what follows, to simplify the statements, we shall assume that $\dim X < \infty$, wherefore $X_{0\mathbf{D}}$ is always closed; we shall denote by $X_{1\mathbf{D}}$ any complementary subspace of $X_{0\mathbf{D}}$. These restrictions are not essential, except that, if $\dim X = \infty$, it must be assumed that $X_{0\mathbf{D}}$ is closed (but not necessarily that it has a complementary subspace).

THEOREM 3. *If $(\mathbf{B}, \mathbf{D})$ is an admissible $\mathfrak{J}$-pair, there exist positive functions $M_0(\Delta)$, $M_0'(\Delta)$ of $\Delta > 0$, respectively non-increasing and non-decreasing, such that, if $t \ge t_0 \ge 0$,*

(i) *for any solution $x(t)$ of (1) with $x(0) \in X_{0\mathbf{D}}$ (any $\mathbf{D}$-solution)*

$$\int_t^{t+\Delta} \|x(\tau)\|\, d\tau \le M_0(\Delta) \int_{t_0}^{t_0+\Delta} \|x(\tau)\|\, d\tau$$

$$|\chi_{[t,t+\Delta]}\, x|_{\mathbf{D}} \le M_0(\Delta)\, |\chi_{[t_0,t_0+\Delta]}\, x|_{\mathbf{D}},$$

(χ_E *being the characteristic function of the set* $E \subset J$);

(ii) *for any solution* $x(t)$ *of* (1) *with* $x(0) \in X_{1\mathbf{D}}$

$$\int_t^{t+\Delta} \| x(\tau) \| \, d\tau \geq M_0'(\Delta) \int_{t_0}^{t_0+\Delta} \| x(\tau) \| \, d\tau,$$

$$| \chi_{[t,t+\Delta]} \, x |_{\mathbf{D}} \geq M_0'(\Delta) \, | \chi_{[t_0,t_0+\Delta]} \, x |_{\mathbf{D}}.$$

THEOREM 4. *If the* $\mathfrak{Z}$*-pair* $(\mathbf{B}, \mathbf{D})$ *is admissible and not weaker than* $(\mathbf{L}^1, \mathbf{L}_0^\infty)$ ($\mathbf{L}_0^\infty$: *the subspace of* $\mathbf{L}^\infty$ *consisting of the functions* x *with* $\operatorname{ess\,lim}_{t\to\infty} x(t) = 0$), *there exist positive functions* $M(\Delta)$, $M'(\Delta)$, *respectively non-increasing and non-decreasing, and numbers* ν, ν' *such that, if* $t \geq t_0 \geq 0$,

(i) *for any solution* $x(t)$ *of* (1) *with* $x(0) \in X_{0\mathbf{D}}$ (*any* $\mathbf{D}$*-solution*)

$$\int_t^{t+\Delta} \| x(\tau) \| \, d\tau \leq M(\Delta) \, e^{-\nu(t-t_0)} \int_{t_0}^{t_0+\Delta} \| x(\tau) \| \, d\tau,$$

$$| \chi_{[t,t+\Delta]} \, x |_{\mathbf{D}} \leq M(\Delta) \, e^{-\nu(t-t_0)} \, | \chi_{[t_0,t_0+\Delta]} \, x |_{\mathbf{D}};$$

(ii) *for any solution* $x(t)$ *of* (1) *with* $x(0) \in X_{1\mathbf{D}}$,

$$\int_t^{t+\Delta} \| x(\tau) \| \, d\tau \geq M'(\Delta) \, e^{\nu'(t-t_0)} \int_{t_0}^{t_0+\Delta} \| x(\tau) \| \, d\tau,$$

$$| \chi_{[t,t+\Delta]} \, x |_{\mathbf{D}} \geq M'(\Delta) \, e^{\nu'(t-t_0)} \, | \chi_{[t_0,t_0+\Delta]} \, x |_{\mathbf{D}}.$$

The type of behavior of the solutions of (1) which is described by (i), (ii) of Theorems 3 [4] may be conveniently labeled *uniform* [*exponential*] *conditional stability in the mean* (in the case of integrals) or *in slices* (in the other case). In the following two theorems a more precise "pointwise" type of behavior is considered, which is described in the following definitions:

We shall say that two complementary subspaces Y_0, Y_1 of X (the definition can be extended also, if $\dim X = \infty$, to the case where Y_0 has no complementary subspace) induce *a dichotomy of the solutions of* (1) if positive constants N_0, N_0', γ_0 exist such that:

(Di) for any solution $x(t)$ of (1) with $x(0) \in Y_0$, and any $t \geq t_0 \geq 0$, $\| x(t) \| \leq N_0 \, \| x(t_0) \|$;

(Dii) for any solution $x(t)$, $x(0) \in Y_1$, $\| x(t) \| \geq N_0' \, \| x(t_0) \|$, $t \geq t_0 \geq 0$;

(Diii) for any pair of non trivial solutions $x_0(t)$, $x_1(t)$, $x_i(0) \in Y_i$, $i = 0, 1$, $\gamma[x_0(t), x_1(t)] \geq \gamma_0$, $t \geq 0$ (where $\gamma[x, y] = \| \, x \, \| \, x \, \|^{-1} - y \, \| \, y \, \|^{-1} \, \|$ is the "angular distance", cf. [1]).

We shall say that Y_0, Y_1 induce *an exponential dichotomy of the solutions of* (1) if positive constants N, N', ν, ν', γ_0 exist such that:

(Ei) for any solution $x(t)$ of (1) with $x(0) \in Y_0$, and any $t \geq t_0 \geq 0$, $\| x(t) \| \leq N e^{\nu(t-t_0)} \, \| x(t_0) \|$;

(Eii) for any solution $x(t)$ of (1) with $x(0) \in Y_1$, and any $t \geq t_0 \geq 0$, $\| x(t) \| \geq N' e^{\nu'(t-t_0)} \, \| x(t_0) \|$;

(Eiii) = (Diii).

In the particular case $Y_0 = X$, the existence of a [exponential] dichotomy is equivalent to uniform [exponential or asymptotic, both things being equivalent for linear systems] stability. In the general case, this type of behavior may be termed uniform conditional [exponential] stability.

THEOREM 5. *If the* $\mathfrak{J}$*-pair* $(\mathbf{B}, \mathbf{D})$ *is admissible and, moreover, is stronger than either* $(\mathbf{L}^1, \mathbf{L}^\infty)$ *or* $A \in \mathbf{M}$, *the subspaces* $X_{0\mathbf{D}}$, $X_{1\mathbf{D}}$ *induce a dichotomy of the solutions of* (1). *Conversely, if a dichotomy exists,* $(\mathbf{L}^1, \mathbf{L}_0^\infty)$ *is admissible (and any weaker pair).*

THEOREM 6. *If the* $\mathfrak{J}$*-pair* $(\mathbf{B}, \mathbf{D})$ *is admissible and not weaker than* $(\mathbf{L}^1, \mathbf{L}_0^\infty)$ *and if a dichotomy exists (in particular, if* $A \in \mathbf{M}$*), the subspaces* $X_{0\mathbf{D}}$, $X_{1\mathbf{D}}$ *induce an exponential dichotomy of the solutions of* (1). *Conversely, if an exponential dichotomy exists, the following pairs are admissible (among many other "strong" pairs):* $(\mathbf{M}, \mathbf{L}^\infty)$, $(\mathbf{M}_0, \mathbf{L}_0^\infty)$, $(\mathbf{L}^1, \mathbf{T})$ *(*$\mathbf{M}_0$ *: the subspace of* $\mathbf{M}$ *consisting of the functions* φ *with* $\lim_{t\to\infty} \int_t^{t+1} |\varphi(\tau)| \, d\tau = 0$*).*

Section 4

The following selection of theorems may give an idea of other problems which have been investigated with these methods:

a) Theorems on the existence of almost-periodic solutions (essentially in [6], [10]):

THEOREM 7. *If the* $\mathfrak{J}$*-pair* $(\mathbf{B}, \mathbf{D})$ *is admissible and not weaker than* $(\mathbf{L}^1, \mathbf{L}_0^\infty)$ *and if* $A(t)$ *is almost-periodic, for each almost-periodic* f *equation* (2) *has one and only one almost-periodic solution.*

b) Theorems on non-linear equations (essentially in [6]):

THEOREM 8. *Let* $(\mathbf{B}, \mathbf{D})$ *be an admissible* $\mathfrak{J}$*-pair and* $h(x, t)$ *a function defined for* $t \in J$, $x \in X$, $\| x \| < a$ $(0 < a \leq \infty)$, *with values in* X, *such that for any function* $x \in \mathbf{D} \cap \mathbf{L}^\infty$, $| x |_\infty < a$, *we have* $h(x(t), t) \in \mathbf{B}$. *Let* $\beta = | h(0, t) |_{\mathbf{B}}$ *and assume that there exists a positive constant* λ *such that for any pair* x', $x'' \in \mathbf{D} \cap \mathbf{L}^\infty$, $| x' |_\infty$, $| x'' |_\infty < a$, *we have*

$$| h(x'(t), t) - h(x''(t), t) |_{\mathbf{B}} \leq \lambda | x' - x'' |_{\mathbf{D}} .$$

Then, if β, λ *are small enough, a positive number* b *exists such that for any* $\xi_0 \in X_{0\mathbf{D}}$, $\| \xi_0 \| < b$, *there is one and only one solution* $x(t, \xi_0) \in \mathbf{D}$ *of* (3) *such that* $x(0, \xi_0) = \xi_0 + \xi_1$, $\xi_1 \in X_{1\mathbf{D}}$.

COROLLARY. *Let* $(\mathbf{B}, \mathbf{D})$ *be an admissible* $\mathfrak{J}$*-pair, not weaker than* $(\mathbf{L}^1, \mathbf{L}_0^\infty)$, *and assume* $A \in \mathbf{M}$. *Let* $h(x, t)$ *be defined for* $t \in J, x \in X$, $\| x \| < a$; *assume that, for each fixed* x *it is a bounded continuous function of* t *and that there is a positive constant* λ *such that* $\| h(x', t) - h(x'', t) \| \leq \lambda \| x' - x'' \|$ *for any* x', $x'' \in X$, $\| x' \|$, $\| x'' \| < a$, $t \geq 0$. *Then the conclusion of Theorem* 8 *holds, with* $\mathbf{D}$ *replaced by* $\mathbf{C}$.

THEOREM 9. *Assume that* X *is reflexive and that an exponential dichotomy of the solutions of* (1) *exists. Let* $h(x, t)$ *be a weakly continuous function from* $X \times J$

into X *(i.e. a function which is continuous when* X *is endowed with the weak topology); assume that* $\| h(x, t) \| \leq \varphi(\| x \|)$, $\varphi(r)$ *being a function defined and continuous for* $r \geq 0$, $\varphi(r) = o(r)$ *as* $r \to \infty$. *Then, for each* $\xi_0 \in X_{0C}$, *equation* (3) *has at least one bounded solution* $x(t, \xi_0)$ *such that* $x(0, \xi_0) = \xi_0 + \xi_1$, $\xi_1 \in X_{1C}$.

Similar results hold for the existence of almost-periodic solutions of (3), cf. [6], [10].

c) "Roughness" theorems (essentially in [6]):

THEOREM 10. *Let* $(\mathbf{B}, \mathbf{L}^\infty)$ *be admissible and* $B \in \mathbf{B}(\tilde{X})$ ($\tilde{X}$: *the space of endomorphisms of* X). *Then, if* $| B |_{\mathbf{B}}$ *is small enough,* $(\mathbf{B}, \mathbf{L}^\infty)$ *is admissible for the equation* $\dot{x} + (A(t) + B(t))x = f(t)$.

d) Theorems on Lyapunov's second method [8]:

THEOREM 11. *A necessary and sufficient condition for the existence of an exponential dichotomy is that a (generalized) Lyapunov function* V *exist with an infinitely small upper bound and such that* V' *(the total derivative by virtue of* (1)) *is a definite function.*

THEOREM 12. *A necessary and sufficient condition for the existence of a dichotomy is that two non-negative functions* V_0, V_1 *exist, which are positively homogeneous of the same degree and such that* $V_0 + V_1$ *is positive definite and has an infinitely small upper bound and* $V_0' \leq 0$, $V_1' \geq 0$.

e) Theorems on periodic equations (which are essentially new only if $\dim X = \infty$) [7]:

THEOREM 13. *If* $A(t)$ *is periodic of period* 1 *and* $| A |_{\mathbf{M}} < \log 4$, *there exists a Floquet representation of the solutions of* (1): $x(t) = P(t)e^{tB}x_0$, *with* P *periodic of period* 1, B *constant.*

Such a representation is not always possible; there are counterexamples with $| A |_{\mathbf{M}}$ exceeding π by an arbitrarily small number.

THEOREM 14. *Let* $A(t)$ *be periodic of period* 1 *and assume that the closure of* X_{0C} *is reflexive. If, for some* f *periodic of period* 1, *equation* (2) *has a bounded solution, it has a periodic solution of period* 1.

THEOREM 15. *If* $(\mathbf{B}, \mathbf{D})$ *is an admissible* $\mathfrak{J}$*-pair, not weaker than* $(\mathbf{L}^1, \mathbf{L}_0^\infty)$, *and if* $A(t), f(t)$ *are periodic of period* 1, *there is one and only one solution of* (2) *which is periodic of period* 1.

f) Theorems on equations with constant coefficients (generalizations to the infinite-dimensional case) [13].

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BIBLIOGRAPHY

[1] J. A. CLARKSON, *Uniformly convex spaces*, Trans. Amer. Math. Soc., 40 (1936), 396–414.
[2] M. G. KREĬN, *On some problems related to the ideas of Lyapunov in the theory of stability*, Uspehi Mat. Nauk (N. S.), 3, N° 3 (25) (1948), 166–169 (Russian).

[3] D. L. Kučer, *On some criteria of boundedness of the solutions of a system of differential equations*, Doklady Akad. Nauk SSSR, 69 (1949), 603–606 (Russian).
[4] A. D. Maĭzel', *On the stability of the solutions of systems of differential equations*, Trudy Ural'skogo Politehn. Inst., 5 (1954), 20–50 (Russian).
[5] I. G. Malkin, *On stability in the first approximation*, Sbornik Naučnyh Trudov Kazanskogo Aviacionnogo Inst., 3 (1935), 7–17 (Russian).
[6] J. L. Massera and J. J. Schäffer, *Linear differential equations and functional analysis, I*, Annals of Math., 67 (1958), 517–573.
[7] J. L. Massera and J. J. Schäffer, *Linear differential equations and functional analysis, II. Equations with periodic coefficients*, ibid, 69 (1959), 88–104.
[8] J. L. Massera and J. J. Schäffer, *Linear differential equations and functional analysis, III. Lyapunov's second method in the case of conditional stability*, ibid, 69 (1959), 535–574.
[9] J. L. Massera and J. J. Schäffer, *Linear differential equations and functional analysis, IV*, Math. Annalen, 139 (1960), 287–342.
[10] J. L. Massera, *Un criterio de existencia de soluciones casi-periódicas de ciertos sistemas de ecuaciones diferenciales casi-periódicas*, Publ. Inst. Mat. Estad. (Montevideo), III (1958), 99–103.
[11] O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Zeitschrift, 32 (1930), 703–728.
[12] K. P. Persidskiĭ, *On the stability of motion in the first approximation*, Mat. Sbornik, (N. S.) 40 (1933), 284–293 (Russian).
[13] J. J. Schäffer, *Ecuaciones diferenciales lineales con coeficientes constantes en espacios de Banach*, Publ. Inst. Mat. Estad. (Montevideo), III (1958), 105–110.
[14] J. J. Schäffer, *Function spaces with translations*, Math. Annalen, 137 (1959), 209–262.