ON THE METHOD OF KRYLOV-BOGOLIUBOV-MITROPOLSKI FOR THE EXISTENCE OF INTEGRAL MANIFOLDS OF PERTURBED DIFFERENTIAL SYSTEMS

By Jack K. Hale*

Section 1

The purpose of the present note is to point out some important results of N. Bogoliubov and Y. Mitropolski ([1]) (which seem not to be well known in this country) on the existence of integral manifolds of differential systems. The results in [1] are stated in terms of a particular problem, but the proofs of these results apply to a much larger variety of questions. In this paper, we shall try to mention only those generalizations which are essential to make an association with some recent results of various authors in this country, and postpone a more complete discussion of the generalizations to a later paper ([5]) In particular, we will mention some aspects of the work of N. Levinson ([9]), S. P. Diliberto and his colleagues ([2], [3], [4], [6], [7], [9], [11], [12]). We begin by discussing two types of problems, one considered by N. Krylov and N. Bogoliubov ([8]) and the other considered by N. Levinson ([10]).

Section 2

The motivation for the discussion in [1] stemmed from a paper in 1934 by N. Krylov and N. Bogoliubov [8] on the "method of averaging" as applied to systems of weakly nonlinear differential systems. Consider the system of equations

$$(1) \qquad \frac{d^2 z_j}{dt^2} + \omega_j^2 z_j = \epsilon Z_j\left(t, z_1, \cdots, z_n, \frac{dz_1}{dt}, \cdots, \frac{dz_n}{dt}\right), \quad j = 1, 2, \cdots, n.$$

By a simple calculation, the transformation of variables,

$$z_j = x_{2j-1} \exp\,(i\omega_j t) + x_{2j} \exp\,(-i\omega_j t),$$

$dz_j/dt = i\omega_j x_{2j-1} \exp\,(i\omega_j t) - i\omega_j \exp\,(-i\omega_j t)$, $j = 1, 2, \cdots, n$, where x_{2j-1}, x_{2j} are complex conjugate, leads to an equivalent system of equations, $dx_{2j-1}/dt = +\epsilon(2i\omega_j)^{-1}Z_j \exp\,(i\omega_j t) \equiv \epsilon X_{2j-1}(t, x_1, \cdots, x_{2n})$, $dx_{2j}/dt = -\epsilon(2i\omega_j)^{-1}Z_j \exp\,(i\omega_j t) \equiv \epsilon X_{2j}(t, x_1, \cdots, x_{2n})$, $j = 1, 2, \cdots, n$. The functions X_j could be very complicated functions of t. Even in the case where the original Z_j are independent of t, the functions X_j are generally almost periodic in t with basic frequencies $\omega_1, \cdots, \omega_n$.

By the above simple transformation, one can therefore always discuss the behavior of the solutions of a system of the form (1) by discussing the be-

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havior of general systems of equations of the form $dx/dt = \epsilon X(t, x)$, where x, X are m-vectors. We may also assume $X(t, x)$ is real. The Krylov-Bogoliubov-Mitropolski method of averaging consists in first supposing that $X_0(x) = \lim_{T\to\infty} T^{-1}\int_0^T X(t, x)\, dt$ exists and then considering the equation $dx/dt = \epsilon X_0(x)$ as the equation of the first approximation to a solution of the original equation. The important question is the following: Under what conditions are the qualitative behavior of the solutions of the first approximation and the original equation the same? This is partially answered by the following theorems of Bogoliubov and Mitropolski [1]. A vector function $f(t, x)$ is said to be *almost periodic in t uniformly with respect to x* in a set Λ, if for any $\eta > 0$, it is possible to find an $l(\eta)$ such that in any interval R, of length $l(\eta)$, there is a τ, $l(\eta)$ and τ independent of x, such that the inequality $\| f(t + \tau, x) - f(t, x) \| \leq \eta$ is satisfied for all real t and $x \in \Lambda$.

THEOREM 1. Suppose the function $X(t, x)$ contained in the equation,

$$\frac{dx}{dt} = \epsilon X(t, x), \tag{2}$$

where x, X are n-vectors, satisfies the following conditions for x contained in some open set U^n:

a) $X(t, x)$ is an almost periodic function of t uniformly with respect to x;

b) $X(t, x)$ and its partial derivatives of first order with respect to x are bounded and uniformly continuous for $-\infty < t < +\infty$, $x \in U^n$. Also, suppose that the equation of the first approximation

$$\frac{dx}{dt} = \epsilon X_0(x), \tag{3}$$

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t, x)\, dt, \tag{4}$$

has a constant solution x^0 and the real parts of the characteristic roots of the linear variational equation of x^0 are different from zero. Under these conditions it is possible to find positive numbers ϵ_0, σ_0 such that for every ϵ, $0 < \epsilon < \epsilon_0$, the following conclusions hold:

i) Equation (2) has a unique solution, $x = x^\star(t)$, which is defined in $(-\infty, +\infty)$ and for which $\| x^\star(t) - x^0 \| \leq \sigma_0$; moreover $x^\star(t)$ is almost periodic with the same basic frequencies as the function $X(t, x)$; and, further, $\| x^\star(t) - x^0 \| \leq \delta(\epsilon)$, $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, $-\infty < t < +\infty$;

ii) The stability properties of the solution x^0 of (2) are the same as the stability properties of the solution x^0 of (3).

THEOREM 2. Suppose that system (3) has a periodic solution $x^0(\epsilon\omega t)$, $x^0(s) = x^0(s + \pi)$ and the real parts of $n - 1$ of the characteristic exponents of the linear variational equation of $x^0(\epsilon\omega t)$ are different from zero. Also, suppose that the function $X(t, x)$ satisfies condition a) of Theorem 1 in a ρ-neighborhood,

U_ρ, of the periodic solution x^0 and has partial derivatives with respect to x of the second order which are bounded and uniformly continuous with respect to x for $-\infty < t < +\infty$, $x \in U_\rho$. Under these conditions it is possible to find positive numbers ϵ_0, σ_0, $\sigma_0 < \rho$, such that for every ϵ, $0 < \epsilon < \epsilon_0$, the following conclusions hold:

i) Equation (2) has a unique integral manifold S, lying for all real t in U_{σ_0}, having a parametric representation,

$$x = f(t, \theta, \epsilon), \tag{5}$$

defined for all real t, θ, periodic in θ of period π and almost periodic in t uniformly with respect to θ with the same basic frequencies as those of $X(t, x)$; and $\| f(t, \theta, \epsilon) - x^0(\theta) \| \le \delta(\epsilon)$, $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, $-\infty < t < +\infty$, $-\infty < \theta < +\infty$. Also, $f(t, \theta, \epsilon)$ has uniformly continuous derivatives with respect to θ up through second order. Furthermore, there is a function $F(t, \theta, \epsilon)$, defined for all real t, θ, periodic in θ of period π and almost periodic in t uniformly with respect to θ possessing continuous derivatives through the second order such that equation (2) is equivalent to the equation

$$\frac{d\theta}{dt} = \epsilon F(t, \theta, \epsilon) \tag{6}$$

on the manifold S.

ii) If the cylinder of periodic solutions $x^0(\epsilon\omega t + \varphi)$, φ an arbitrary constant, of (3) is stable, unstable, or conditionally stable with respect to a manifold of dimension s, then the integral manifold S of (2) is stable, unstable, or conditionally stable with respect to a manifold of dimension s.*

Let us briefly summarize the method of proof of Theorem 2 used by Bogoliubov and Mitropolski. Since $m - 1$ of the characteristic exponents of the linear variational equation with respect to $x = x^0(\epsilon\omega t)$ have nonzero real parts, there exists a transformation of variables $x \to (s, h)$, s a scalar, h an $(m - 1)$-vector, such that (2) is equivalent to a system of the form

$$\begin{aligned} \frac{ds}{dt} &= \epsilon\omega + \epsilon S_1(s, h) + \epsilon S_2(t, s, h) \\ \frac{dh}{dt} &= \epsilon H h + \epsilon H_1(s, h) + \epsilon H_2(t, s, h) \end{aligned} \tag{7}$$

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* Suppose $dx/dt = X(t, x)$ is a nth order differential system and suppose that S is a manifold of solutions of this equation of dimension m in the $(n + 1)$-dimensional (x, t)-space. The manifold S will be said to be stable if there exist neighborhoods U_1, U_2 of dimensions $n + 1$ of the manifold S, $U_1 \subset U_2$, such that for every solution $x(t)$ of the above equation with $x(t_0) \in U_1$, we have $x(t) \in U_2$ for all $t \ge t_0$ and $x(t) \to S$ as $t \to +\infty$. If for every $x(t)$ with $x(t_0) \in U_1 - S$, we have $x(t) \not\to S$ as $t \to \infty$, then S will be said to be unstable. If U_1 is of dimension $s < n + 1$, and if for every $x(t)$ with $x(t_0) \in U_1$ we have $x(t) \in U_2$ for $t > t_0$ and $x(t) \to S$ as $t \to \infty$, and $x(t_0) \in U_2 - U_1$ implies $x(t) \not\to S$ as $t \to \infty$, then S will be said to be conditionally stable with respect to a manifold of dimension s.

for $\| h \| \leq \sigma$, $\sigma > 0$, where the eigenvalues of the constant matrix H are the nonzero characteristic exponents of the linear variational equation. Furthermore, the vectors S_j, H_j, are periodic in s of period π, $S_1 = 0(\| h \|^2)$, $H_1 = 0(\| h \|^2)$ as $\| h \| \to 0$, S_2, H_2 are almost periodic in t, uniformly with respect to s, h, $-\infty < s < +\infty$, $\| h \| \leq \sigma$, and satisfy the relation

$$(8) \qquad \lim_{T\to\infty} \frac{1}{T} \int_0^T S_2(t, s, h)\, dt = 0, \quad \lim_{T\to\infty} \int_0^T H_2(t, s, h)\, dt = 0.$$

In these new coordinates (s, h), the periodic solution $x^0(\epsilon\omega t + \varphi)$ of (3) is given by $h = 0$, $s = \epsilon\omega t + \varphi$.

Now the proof of Theorem 2 will be complete if it can be shown that there exists for ϵ sufficiently small, a unique manifold of solutions of (7) of the form $h = f(t, \theta, \epsilon)$ where $f(t, \theta, \epsilon)$ is periodic in θ of period π, almost periodic in t uniformly with respect to θ, ϵ, approaches zero uniformly as $\epsilon \to 0$ and has certain stability properties. One of the principal difficulties lies in the fact that the terms ϵS_2, ϵH_2 are not small (i.e., higher order in ϵ or higher order in h) compared with the terms $\epsilon\omega$ and ϵHh.

The next step of the proof consists in finding functions $u(t, s, h, \epsilon)$, $v(t, s, h, \epsilon)$ such that the transformation of variables, $s = \theta + \epsilon u(t, \theta, z, \epsilon)$, $h = z + \epsilon v(t, \theta, z, \epsilon)$, transforms (7) into an equivalent system

$$(9) \qquad \begin{aligned} \frac{d\theta}{dt} &= \epsilon\omega + \epsilon\Theta(t, \theta, z, \epsilon), \\ \frac{dz}{dt} &= \epsilon Hz + \epsilon Z(t, \theta, z, \epsilon), \end{aligned}$$

where Θ, Z are periodic in θ of period π, and $\Theta = 0(| \epsilon | + (| \epsilon | + \| z \|)^2)$ $Z = 0(| \epsilon | + (| \epsilon | + \| z \|)^2)$ as $| \epsilon | \to 0$, $\| z \| \to 0$; i.e., the functions Θ, Z are small as compared to ω, Hz when $| \epsilon |$, $\| z \|$ are small. Now, the change of variable $\tau = \epsilon t$ yields the equation

$$(10) \qquad \begin{aligned} \frac{d\theta}{d\tau} &= \omega + \Theta(\tau, \theta, z, \epsilon), \\ \frac{dz}{d\tau} &= Hz + Z(\tau, \theta, z, \epsilon). \end{aligned}$$

It is then shown that this equation has an integral manifold of the form $z = f(t, \theta, \epsilon)$, $f(t, \theta, \epsilon)$ periodic in θ of period π and almost periodic in t uniformly with respect to θ, ϵ and

$$(11) \qquad \begin{aligned} &\| f(t, \theta, \epsilon) \| \leq D(\epsilon), \\ &\| f(t, \theta_1, \epsilon) - f(t, \theta_2, \epsilon) \| \leq \Delta(\epsilon) | \theta_1 - \theta_2 | \end{aligned}$$

for all t, θ, θ_1, θ_2 and $D(\epsilon)$, $\Delta(\epsilon) \to 0$ as $\epsilon \to 0$. The stability properties of this manifold are established, thus proving Theorem 2. It is probably instructive to review the proof of these last results.

The idea is to consider a class of "generalized cylinders" $\mathcal{C}(D, \Delta) = \{F(t, \theta, \epsilon) \mid F$ is an $(m - 1)$-vector, $F(t, \theta, \epsilon) = F(t, \theta + \pi, \epsilon)$, $\| F(t, \theta, \epsilon) \| \leq D$, $\| F(t, \theta_1, \epsilon) - F(t, \theta_2, \epsilon) \| \leq \Delta | \theta_1 - \theta_2 |$ for all $t, \theta, \theta_1, \theta_2$, $0 < \epsilon \leq \epsilon_0\}$. A transformation, obtained from the differential system, is defined which maps each member of $\mathcal{C}(D, \Delta)$ into a member of $\mathcal{C}(D, \Delta)$ and by choosing D, Δ as convenient functions of ϵ, this map is a contraction. Therefore, it has a fixed point, which is an integral manifold of (10). The almost periodicity in t and the stability properties are then deduced from the fact that this fixed point satisfies the differential equation. One interesting thing to notice is that since the map is a contraction, the integral manifold can be calculated by successive approximations.

Section 3

Let us now turn to a consideration of the problem of N. Levinson ([10]). Consider the vector differential system

$$x' = X(x) + \epsilon X(t, x), \tag{12}$$

where ϵ is a real parameter and $X(x)$, $X(t, x)$ are sufficiently smooth. In 1951, N. Levinson discussed system (12) for the case where $X(t + \pi, x) = X(t, x)$ and showed that if (12) for $\epsilon = 0$ has an asymptotically orbitally stable periodic solution ($m - 1$ of the characteristic exponents of the linear variational equation have negative real parts), then (12) for $\epsilon \neq 0$, but sufficiently small, has a two dimensional torus of solutions which is asymptotically stable. Let us briefly review the method used by Levinson to prove the above results.

For $\epsilon = 0$ in (12) the periodic solution is a limit cycle, C, in x-space. In (x, t)-space the limit cycle becomes a cylinder parallel to the t axis and the cylinder is generated by all solutions of (12) for $\epsilon = 0$ which start at $t = 0$ on the limit cycle and are regarded as curves in (x, t)-space. Since this cylinder is stable, it seems plausible for $\epsilon \neq 0$ and sufficiently small that there will be a unique cylinder, S, depending on ϵ, and equal to the original one for $\epsilon = 0$ which is stable. Furthermore, if $X(t, x)$ is periodic in t of period T, then one suspects that the cross sections of S at $t = 0$ and $t = T$ should be congruent. Because of the periodicity of $X(t, x)$, one can, therefore, identify the cross sections to obtain a torus.

This last remark is the motivation for the technique of proof used by Levinson. More specifically, suppose F is a family of closed curves in x-space which are "very close" to the limit cycle, C. A transformation V, is then defined which maps each member of F into a member of F, namely, V of a point P is $x(T)$ where $x(t)$ is a solution of (12) with $x(0)$ given by P. It is then shown that V has a fixed point in F.

This theorem follows very easily from the results mentioned in section 2. In fact, by introducing new coordinates (θ, h), θ a scalar, h an $(m - 1)$ vector, in a neighborhood of the limit cycle, C, as in the discussion preceding formulas 9 and 10, one obtains a system of equations of the form (10) with the eigenvalues of H having negative real parts and Θ, Z periodic in t. The above result of Levinson then follows immediately and one can prove even more. In fact, the function $X(t, x)$ need not be periodic in t, but may be almost periodic in t uniformly with respect to x and the eigenvalues of H need only have nonzero real parts. It has been shown ([16]) that the method of Levinson can be extended to the case where H has eigenvalues with nonzero real parts, but the method of proof does not seem to be generalizable to the case of arbitrary almost periodic functions of t.

Recently, using the geometric idea of the proof of Levinson as a starting point, S. P. Diliberto and his colleagues ([2], [3], [4], [6], [7], [9], [11], [12]) have been studying a more general perturbation theory where the objects of interest are not necessarily surfaces which are homeomorphic to the topological product of two circles as in Levinson's paper, but are homeomorphic to the topological product of k circles (periodic surfaces). The systems that have been discussed thus far by Diliberto, et. al., are of the form

$$\begin{aligned} \frac{d\theta}{dt} &= d + \epsilon[d(t, \theta) + \Theta^{\star}(t, \theta, y, z, \epsilon)] \\ \frac{dy}{dt} &= \epsilon a(t, \theta) + \epsilon C(t, \theta)y + \epsilon Y(t, \theta, y, z, \epsilon) \qquad (13) \\ \frac{dz}{dt} &= A(t, \theta)z + Z(t, \theta, y, z, \epsilon) \end{aligned}$$

where θ, y, z are vectors; all of the functions are periodic in t, θ; and the functions $\Theta^{\star}$, Y, Z satisfy the smallness conditions imposed on Θ, Y in (10) when $|\epsilon|$, $\|y\|$, $\|z\|$ are small. System (13) is much more general then the system (10) in the sense that the vector equation in y and the functions $d(t, \theta)$, $a(t, \theta)$ are introduced and the matrices $C(t, \theta)$, $A(t, \theta)$ are not constant. However, all functions are periodic in t. In case $a(t, \theta) = 0 = d(t, \theta)$ and $C(t, \theta)$, $A(t, \theta)$ are constant matrices, it is not very difficult to generalize the results of N. Bogoliubov and Y. Mitropolski to systems of the type (13), thus eliminating the restriction that the functions be periodic in t. When this is done one obtains a generalization of the result of W. T. Kyner ([9]). This generalization will appear in [5].

If $f(s)$ is periodic in the vector s with a period ω, then define $\bar{f}(s)$ by the relation $\bar{f}(s) = \lim_{T\to\infty} T^{-1}\int_0^T f(s + t)\,dt$. In [2], Diliberto has discussed general systems of the type (13) under various conditions on the functions $\bar{a}(t, \theta)$, $\bar{d}(t, \theta)$, $\bar{C}(t, \theta)$, $\bar{A}(t, \theta)$ and shown the existence of periodic surfaces. It is important in the applications to consider these more general systems as one can easily see by studying the original paper of N. Krylov and N. Bogoliubov [8]. In [5] the author discusses this more general situation along the lines of Bogoliubov and Mitropolski and, therefore, allows the functions in (13) to be almost periodic in t. However, we do not discuss this any further here since it would take us too far afield.

RIAS, Baltimore, Maryland

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