ON THE CHARACTERISTIC EXPONENTS OF LINEAR PERIODIC DIFFERENTIAL SYSTEMS

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1. Introduction and summary of results

The purpose of the present note is to point out some of the results that can be obtained by applying the method of successive approximations of L. Cesari, J. K. Hale and R. A. Gambill to the determination of the characteristic exponents of a special linear differential system with periodic coefficients. See the previous paper of L. Cesari, [6], for a description of this method, where also a topological interpretation is given in terms of a fixed point of an integral operator. Also, the reader should consult [5] for a complete list of references on the implications of the method. In this note, we will also state some applications to the stability of periodic solutions of weakly nonlinear differential systems and to the boundedness of the solutions of linear differential systems. For the proofs and a number of further applications, see [11] and [12]. At the same time, we prove by a straightforward argument some results on the boundedness of solutions of linear periodic differential systems which had been previously obtained by the method mentioned above. This type of argument does not seem to be too well known in this country and, therefore, we think it is appropriate to include it here and, at the same time, it may help to clarify why a method of successive approximations may be useful for obtaining new boundedness theorems.

Consider a linear system of differential equations

$$y' = Ay + \epsilon C(t)y, \qquad y = (y_1, \cdots, y_N), \tag{1.1}$$

where ϵ is real, A, C are real matrices, A constant, $C(t + T) = C(t)$, $T = 2\pi/\omega$, $\omega > 0$, $C(t)$ is L-integrable in $[0, T]$, and the eigenvalues of A have simple elementary divisors. In the following we only speak of the absolutely continuous solutions of (1.1). If $Y(t, \epsilon)$ is a fundamental system of real solutions of (1.1) with $Y(0, \epsilon) = I$, then the *characteristic multipliers*, $\rho_j = \rho_j(\epsilon)$, $j = 1, 2, \cdots, N$, of (1.1) are defined as the roots of the equation

$$\det [Y(t_0 + T, \epsilon) - \rho Y(t_0, \epsilon)] = 0 \tag{1.2}$$

for any fixed t_0. It is known ([13]) that the roots of this equation are independent of t_0 and, in particular, for $t_0 = 0$,

$$\det [Y(T, \epsilon) - \rho I] = 0. \tag{1.3}$$

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The *characteristic exponents*, $\tau_j = \tau_j(\epsilon)$, $j = 1, 2, \cdots, N$, determined only up to a multiple of ωi, are defined by

$$\rho_j = e^{\tau_j T}, \qquad j = 1, 2, \cdots, N. \tag{1.4}$$

The roots of (1.2) occur in complex conjugate pairs and, furthermore, each $\rho_j(\epsilon)$ is a continuous function of ϵ at $\epsilon = 0$. In addition, the $\rho_j(0)$ are the roots of the equation

$$\det [e^{AT} - \rho I] = 0 \tag{1.5}$$

Consequently, we know beforehand the multiplicity of a root $\rho_j(0)$. Let $\overset{*}{\rho_k}$, of multiplicity n_k, $k = 1, 2, \cdots, \nu$, $\nu \leq N$, be the distinct roots of (1.5). Since the $\rho_j(\epsilon)$ are continuous in ϵ at $\epsilon = 0$, it follows that there exist ϵ_0, δ such that in a δ-sphere about $\overset{*}{\rho_k}$, there are exactly n_k roots of (1.3) for $0 \leq |\epsilon| \leq \epsilon_0$.

Even with this more detailed information, it does not seem possible to apply directly the Floquet theory to reduce the order of the $N \times N$ matrix in (1.3).

By using the method of successive approximations of Cesari, Gambill and Hale, one can show that it is possible to determine the $\rho_j(\epsilon)$ in groups of n_k at a time; i.e., n_k of the $\rho_j(\epsilon)$, namely, those for which $\rho_j(0) = \overset{*}{\rho_k}$, can be determined independently of the others. More specifically, if $\rho_{j_l}(0) = \overset{*}{\rho_k}$, $l = 1, 2, \cdots, n_k$, then a matrix B_k of the dimension $n_k \times n_k$ is explicitly given whose eigenvalues determine the characteristic exponents τ_{j_l}, $l = 1, 2, \cdots, n_k$, associated with ρ_{j_l}. In particular, if $\rho_j(0)$ is a simple root of (1.3), then the corresponding characteristic exponent is determined directly independently of the other characteristic exponents by the method of successive approximations mentioned above.

There are two main types of problems associated with linear periodic systems (1.1). First, there is the case in which the characteristic multipliers $\rho_j(\epsilon)$ have modulus different from unity and then the solutions of (1.1) will be either unbounded or approach zero as $t \rightarrow \infty$, but no solution will be bounded in $(-\infty, +\infty)$. For this case, the problem of stability can be decided by a finite number of successive approximations using the above method. On the other hand, if the $\rho_j(\epsilon)$ have modulus one, i.e., the solutions are bounded in $(-\infty, +\infty)$, then all the characteristic exponents are purely imaginary and all of the successive approximations must possess some special properties. L. Cesari ([4]) gave the first such boundedness theorems by using the above method of approximations. In Section 2, we discuss further boundedness theorems ($|\rho_j(\epsilon)| = 1$) and in Section 3, we give some theorems which assure that the $\rho_j(\epsilon)$ have modulus less than one.

2. Boundedness of solutions of linear periodic systems

As mentioned in Section 1, we will prove some boundedness theorems without using successive approximations and hope that it will clarify why successive approximations are ultimately used and why certain classes of functions are introduced.

We will actually consider the more general system

$$y' = P(t, \epsilon)y, \quad 0 \leq t < +\infty, \tag{2.1}$$

where $P(t + T, \epsilon) = P(t, \epsilon), T = 2\pi/\omega, \omega > 0$, for every real $\epsilon, 0 \leq |\epsilon| \leq \epsilon_0$, $\epsilon_0 > 0$, is a matrix of order N whose elements $p_{jk}(t, \epsilon)$, $j, k = 1, 2, \cdots, N$, are real functions of the real variables t, ϵ, $|p_{jk}(t, \epsilon)| < \eta(t)$ for all $j, k, 0 \leq |\epsilon| \leq \epsilon_0$, where $\eta(t)$ is L-integrable in $[0, T]$ and each $p_{jk}(t, \epsilon)$ is a continuous function of ϵ at $\epsilon = 0$. The characteristic multipliers and exponents are defined exactly as in Section 1 and it is known for this more general situation that the multipliers are continuous functions of ϵ at $\epsilon = 0$. If for each ρ which is a root of (1.3), ρ^{-1} is also a root, then we say that system (2.1) is *reciprocal*. The theorems on reciprocal systems below and their proofs are essentially due to A. Liapunov ([13]). The same type of reasoning can also be found in I. G. Malkin, [14], and V. A. Yakubovic, [20].

THEOREM 2.1. If system (2.1) is reciprocal and ρ_0, $|\rho_0| = 1$ is a simple characteristic multiplier of (1.3) for $\epsilon = 0$, then there exists an ϵ_1, $0 < \epsilon_1 \leq \epsilon_0$ such that there is a characteristic multiplier, $\rho(\epsilon)$, of (2.1) with $|\rho(\epsilon)| = 1$, $\rho(0) = \rho_0$, $0 \leq |\epsilon| \leq \epsilon_1$.

Proof. Since ρ_0 is a simple characteristic multiplier, it follows from the continuity in ϵ, that there exists an ϵ_1 such that the characteristic multiplier, $\rho(\epsilon)$, with $\rho(0) = \rho_0$ is simple for $0 \leq |\epsilon| \leq \epsilon_1$. Since system (2.1) is assumed to be reciprocal, it follows that $\rho^{-1}(\epsilon)$ is also a characteristic multiplier. Furthermore, $\bar{\rho}(\epsilon)$ is a characteristic multiplier. The conclusion of the theorem is now obvious.

An immediate consequence of this theorem is

COROLLARY 2.1. If system (2.1) is reciprocal and the characteristic multipliers of (2.1) are distinct and have unit modulus for $\epsilon = 0$, then there exists an ϵ_1, $0 < \epsilon_1 \leq \epsilon_0$ such that the solutions of (2.1) are bounded for $-\infty < t < +\infty$, $0 \leq |\epsilon| \leq \epsilon_1$.

THEOREM 2.2. If there exists a matrix $B = B(\epsilon)$ of order N, continuous in ϵ at $\epsilon = 0$, $|\det B(\epsilon)| \geq \delta > 0$, $0 \leq |\epsilon| \leq \epsilon_0$, such that either of the following conditions are satisfied,

(i) $B(\epsilon)P(t, \epsilon)B^{-1}(\epsilon) = -P(-t, \epsilon), \quad 0 \leq |\epsilon| \leq \epsilon_0, -\infty < t < +\infty,$

(ii) $B(\epsilon)P(t, \epsilon)B^{-1}(\epsilon) = -P^*(t, \epsilon), \quad 0 \leq |\epsilon| \leq \epsilon_0, -\infty < t < +\infty,$
(P^* is the transpose of P)

then system (2.1) is reciprocal.

Proof. Let $Y(t)$, $Y(0) = I$ be a fundamental system of real solutions of (2.1). If $Z(t)$ is an $N \times N$ matrix solution of the adjoint equation

$$Z'(t) = -Z(t)P(t, \epsilon), \qquad Z(0) = Z_0, \tag{2.2}$$

then it is known that $Z(t)Y(t) = Z_0 I = Z_0$ for all t.

(i) If $B(\epsilon)P(t, \epsilon)B^{-1}(\epsilon) = -P(-t, \epsilon)$, then $Z(t) = Y^{-1}(-t)B^{-1}(\epsilon)$ satisfies the adjoint equation (2.2). Therefore, $Y^{-1}(-t)B^{-1}(\epsilon)Y(t) = B^{-1}(\epsilon)$, or $Y(t) = B(\epsilon)Y(-t)B^{-1}(\epsilon)$ and the characteristic equations $\det [Y(T) - \rho I] = 0$ and the equation $\det [I - \lambda^{-1}Y(-T)] = 0$ are the same. But, since $Y(0) = I$, these are two special cases of (1.2) with $t_0 = 0$ and $t_0 = -T$. Therefore, the λ are the reciprocals of the ρ and (i) is proved.

(ii) If $B(\epsilon)P(t, \epsilon)B^{-1}(\epsilon) = -P^*(t, \epsilon)$, then $Z(t) = Y^*(t)B(\epsilon)$ satisfies (2.2) and $Y^*(t)B(\epsilon)Y(t) = B(\epsilon)$; or, $Y^*(t) = B(\epsilon)Y^{-1}(t)B^{-1}(\epsilon)$. Since $Y(t)$ is real, the characteristic equation of $Y^*(T)$ is the same as the characteristic equation of $Y(T)$ and the remainder of the proof of (ii) is obvious.

As an application of the previous results, consider the system of equations

$$x_j'' + \sigma_j^2 x_j = \epsilon \sum_{k=1}^{n} \varphi_{jk}(t)x_k , \qquad j = 1, 2, \cdots , n, \tag{2.3}$$

where $\sigma_j > 0$, $j = 1, 2, \cdots , n$, the $\varphi_{jk}(t)$ are periodic functions of t of period $T = 2\pi/\omega$ and L-integrable in [0, T]. The transformation of variables $x_j = y_j$, $x_j' = y_{n+j}$, $j = 1, 2, \cdots , n$, leads to the equivalent system of equations

$$y' = A(t, \epsilon)y \tag{2.4}$$

where $A(t, \epsilon) = [A_{jk}(t, \epsilon)]$, $j, k = 1, 2$, each $A_{jk}(t, \epsilon)$ is an $n \times n$ matrix, $A_{11} = 0 = A_{22}$, $A_{12} = I$, $A_{21} = -\sigma^2 + \epsilon\Phi(t)$, $\sigma^2 = \text{diag } (\sigma_1^2, \cdots , \sigma_n^2)$, $\Phi(t) = [\varphi_{jk}(t)]$, $j, k = 1, 2, \cdots , n$.

(i′) If the matrix Φ can be partitioned as $\Phi = [\Phi_{jk}(t)]$, $j, k = 1, 2$, where Φ_{11} is an $m \times m$ matrix, $m \leq n$, Φ_{22} is an $(n - m) \times (n - m)$ matrix and $\Phi_{jk}(-t) = (-1)^{j+k}\Phi_{jk}(t)$, $j, k = 1, 2$, then $BA(t, \epsilon) = -A(-t, \epsilon)B$ where $B = \text{diag } (I_1 , -I \;, -I_1 , I_2)$ where I_1 is the $m \times m$ identity matrix and I_2 is the $(n - m) \times {}^2(n - m)$ identity matrix.

(ii′) If Φ is symmetric, then $BA = -A^*B$ where $B = (B_{jk})$, $j, k = 1, 2$, each B_{jk} an $n \times n$ matrix with $B_{11} = 0 = B_{22}$, $B_{12} = I$, $B_{21} = -I$.

Therefore, from Theorem 2.1, equation (2.3) is reciprocal if either condition (i′) or (ii′) is satisfied. An immediate consequence of Theorem 2.1 and Corollary 2.1 are the following results.

THEOREM 2.3. If the matrix $\Phi(t) = (\varphi_{jk}(t))$, $j, k = 1, 2, \cdots , n$, associated with system (2.3) satisfies either (i′) or (ii′) and, in addition,

$$2\sigma_1 \not\equiv 0, \qquad \sigma_1 \pm \sigma_j \not\equiv 0 \pmod{\omega}, \quad j = 2, 3, \cdots , n, \tag{2.5}$$

then there exists an ϵ_1, $0 < \epsilon_1 \leq \epsilon_0$ such that there are two linearly independent solutions of (2.3) bounded for $-\infty < t < \infty$, $0 \leq |\epsilon| \leq \epsilon_1$. More specifically, there are two characteristic exponents of (2.3), $\tau_1(\epsilon)$, $\tau_2(\epsilon)$, continuous in ϵ for $0 \leq |\epsilon| \leq \epsilon_1$, which are purely imaginary and $\tau_1(0) = i\sigma_1$, $\tau_2(0) = -i\sigma_1$.

COROLLARY 2.3. If the matrix $\Phi(t)$ satisfies either (i′) or (ii′) and, in addition,

$$2\sigma_j \not\equiv 0, \qquad \sigma_j \pm \sigma_k \not\equiv 0 \pmod{\omega}, \quad j \neq k, \; j, \; k = 1, \; 2, \cdots , \; n, \tag{2.6}$$

then the solutions of (2.3) are bounded for $-\infty < t < \infty$, $0 \leq |\epsilon| \leq \epsilon_1$.

These results have been obtained by L. Cesari [4] and J. K. Hale [10] by using the same method of successive approximations mentioned above, and later by M. Golomb ([9]) using another method of successive approximations, whereas the above reasoning is very straightforward. R. A. Gambill ([7]) has other results which can be obtained in an elementary manner. M. Golomb ([9]) has given another boundedness theorem for a different class of matrices $\Phi(t)$ which is too complicated to describe here. So far, the present author has not been able to determine whether or not this class of functions leads to a reciprocal characteristic equation. It would be of interest to answer this question.

Now suppose that some of the conditions in (2.6) are not satisfied. Is it true that there are unbounded solutions of (2.3) no matter how small $\epsilon \neq 0$ may be? For the well-known Mathieu equation, $x'' + (\sigma^2 + \epsilon \cos \omega t)x = 0$, it is known that, in the (σ, ϵ) plane, there are unbounded solutions in every neighborhood of the points $(m\omega/2, 0)$, $m = 0, 1, 2, \cdots$.

If the matrix $\Phi(t)$ satisfies condition (ii′), then it was shown by I. M. Gelfand and V. B. Lidskii ([8]) that Corollary 2.3 is true if (2.6) is replaced by the condition $\sigma_j + \sigma_k \not\equiv 0 \pmod{\omega}$, $j, k = 1, 2, \cdots, n$. This same result has also been obtained by J. Moser ([17]), and, more recently, generalized somewhat by V. A. Yakubovic ([21]). The proofs of these authors use very strongly the fact that a fundamental system of solutions of (2.3) with condition (ii′) satisfied belongs to the class of symplectic matrices. A simple reasoning of the type used in the previous pages is not of much assistance since a violation of one of the conditions in (2.6) implies that some of the characteristic multipliers of (2.3) are equal for $\epsilon = 0$, and the method is not delicate enough to distinguish between the multiple roots.

Suppose now that some of the conditions in (2.6) are not satisfied and the matrix $\Phi(t)$ satisfies (i′). Since it was not possible to find any algebraic properties of the fundamental system of solutions of (2.3), we made use of the method of successive approximations mentioned in the introduction to obtain many interesting results (see [12]). As an illustration, consider the simplest case where $\Phi(t)$ satisfies (i′) and

$$\sigma_1 - \sigma_2 = s\omega, \qquad 2\sigma_j \not\equiv 0, \qquad \sigma_j \pm \sigma_k \not\equiv 0 \pmod{\omega}, \qquad j \neq k,\ j, k = 2, 3, \cdots, n, \tag{2.7}$$

where s is an integer or zero. Then it is shown that there exists a function $H(s, \sigma_1)$, which can be written down explicitly in terms of the Fourier coefficients of the matrix $\Phi(t)$, such that the solutions of (2.3) are bounded in a sufficiently small neighborhood of the point $(\sigma_1 - s\omega, 0)$ in the (σ_2, ϵ)-plane if $H(s, \sigma_1) > 0$, and there are unbounded solutions of (2.3) in every neighborhood of the point $(\sigma_1 - s\omega, 0)$ if $H(s, \sigma_1) < 0$. If the matrix $\Phi(t)$ satisfies (i′) and, in addition, is symmetric, then $H(s, \sigma_1) > 0$ for all s, σ_1, which agrees with the general result of I. M. Gelfand and V. B. Lidskii [8] mentioned above. However, examples are given to show that $H(s, \sigma_1)$ may be <0 for some matrices $\Phi(t)$ if Φ satisfies

(i') and is not symmetric. More specifically, if $\Phi(t) = (\varphi_{jk}(t))$, $j, k = 1, 2$, $\varphi_{11} = 2p \cos t$, $\varphi_{12} = 2 \cos t$, $\varphi_{21} = 2q \cos t$, $\varphi_{22} = 2r \cos t$, where p, q, r are constants, and $\sigma_1 = \sigma_2(0)$, σ_1 independent of ϵ, then the solutions are bounded in a neighborhood of the point $(\sigma_1, 0)$ in every neighborhood of this point if $p + r \neq 0$, $q < 0$.

As another application, let us consider the following result which has also been obtained by the above method of successive approximations. (See [23] for a proof and also a more general theorem. We state the simpler form of the theorem for brevity.)

THEOREM 2.4. Consider the system of equations

$$(2.8) \qquad \begin{aligned} x'' + Ax &= \epsilon\Phi(t)x + \epsilon\Psi(t)y \\ y' &= \epsilon P(t)x + \epsilon Q(t)y \end{aligned}$$

where x is an n-vector, y is an k-vector, $A = \text{diag}\,(\sigma_1^2, \cdots, \sigma_n^2)$, Φ, Ψ, P, Q are matrices of appropriate dimensions whose elements are periodic in t of period $T = 2\pi/\omega$ and L-integrable in $[0, T]$. If Φ, P are even in t and Ψ, Q are odd in t and $2\sigma_j \not\equiv 0$, $\sigma_j \pm \sigma_k \not\equiv 0 \pmod{\omega}$ $j \neq k$, $j, k = 1, 2, \cdots, n$, then all solutions of (2.8) are bounded in $(-\infty, +\infty)$ for $|\epsilon|$ sufficiently small.

The interest in this theorem lies in the fact that there are m characteristic exponents equal to zero for $\epsilon = 0$ and, therefore, a simple reasoning of the type used in the previous pages is not applicable.

3. Applications to stability of periodic solutions

In this section we state two theorems which seem to be very important and at the same time very easy to apply. For more results and examples, see [11].

THEOREM 3.1. Consider the autonomous system

$$(3.1) \qquad x'' + Ax = \epsilon f(x, x')$$

where x, f are n vectors, $\epsilon > 0$, $A = \text{diag}\,(\sigma_1^2, \cdots, \sigma_n^2)$, $\sigma_j^2 > 0$, $j = 1, 2, \cdots, n$, and $f \in C^2$. Suppose there is a periodic solution $x_0(\epsilon, t) = x_0(\epsilon, t + 2\pi/\omega)$, $\omega = \sigma_1 + 0(\epsilon)$, of (3.1) of the form $x_0(0, t) = (a \cos \sigma_1 t, 0, \cdots, 0)$ where a is a real number. If

$$(3.2) \qquad 2\sigma_j \not\equiv 0, \qquad \sigma_j \pm \sigma_k \not\equiv 0 \pmod{\sigma_1}, \quad j \neq k, \qquad j,\ k = 2,\ 3, \cdots,\ n,$$

and

$$(3.3) \qquad \int_0^{2\pi} f_{jx_j'}\left[x_0\left(0, \frac{t}{\sigma_1}\right), x_0'\left(0, \frac{t}{\sigma_1}\right)\right] dt < 0, \quad j = 1, 2, \cdots, n,$$

then there exists an $\epsilon_0 > 0$ such that this periodic solution $x_0(\epsilon, t)$ is asymptotically orbitally stable in $[0, \infty)$, $0 < \epsilon \leq \epsilon_0$.

After having the results mentioned in Section 1, the proof of this theorem is quite simple because condition (3.2) implies that the characteristic multipliers for $\epsilon = 0$ of the linear variational equation of $x_0(\epsilon, t)$ are all simple except for two which are equal to unity. Since (3.1) is autonomous, one of the multipliers is unity even for $\epsilon \neq 0$. Consequently, the method of successive approximations essentially gives the remaining characteristic exponents directly and condition (3.3) assures that these exponents have negative real parts.

The above result generalizes a result of A. Andronov and A. Witt ([1]) (see also N. Minorsky [16]), for the case where $n = 2$. They applied the Floquet theory directly and evaluated the corresponding 4×4 determinant, whereas the above proof is much easier and does not require the evaluation of any determinants even for arbitrary n. For arbitrary n, this result was also obtained by E. Thompson ([19]) in his thesis at Purdue University.

THEOREM 3.2. Consider the periodic system

$$x'' + Ax = \epsilon f(x, x', t) \tag{3.4}$$

where x, f are n-vectors, $\epsilon > 0$, $A = \text{diag}\ (\sigma_1^2, \cdots, \sigma_n^2)$, $\sigma_j^2 > 0$, $j = 1, 2, \cdots, n$, $f \in C^2$ with respect to x, x' and continuous in t, $f(x, x', t + T) = f(x, x', t) T = 2\pi/\omega$, $\omega > 0$. Suppose $\sigma_1 = (k/m)\omega$, where k, m are integers, and $m > 0$; and assume that there is a periodic solution $x_0(\epsilon, t)$ of (3.4) of period mT of the form $x_0(0, t) = (a \cos(\sigma_1 t + \varphi), 0, \cdots, 0)$ where a, φ are real numbers. If

$$2\sigma_j \not\equiv 0, \qquad \sigma_j \pm \sigma_k \not\equiv 0 \bmod (\omega/m), \quad j \neq k, \quad j, \ k, = 2, \ 3, \cdots, \ n, \tag{3.5}$$

then there exists an $\epsilon_0 > 0$ such that this periodic solution is asymptotically stable in $[0, \infty)$, $0 < \epsilon \leq \epsilon_0$, provided that

$$\int_0^{mT} f_{jx_j'}\ [x_0\ (0, t), x_0'\ (0, t), t]\ dt < 0, \qquad j = 1, 2, \cdots, n, \tag{3.6}$$

and $A \neq B$, $A^2 \neq B$, $B > 0$, where

$$\begin{aligned} A &= \int_0^{mT} f_{1x_1'}\ dt, \\ 2mT\sigma_1^2\ B &= \sigma_1^2\ A^2 + \left(\int_0^{mT} f_{1x_1}\ dt\right)^2 \\ &\quad - \left[\int_0^{mT} f_{1x_1} \cos 2\sigma_1 t\ dt - \sigma_1 \int_0^{mT} f_{1x_1} \sin 2\sigma_1 t\ dt\right]^2 \\ &\quad - \left[\int_0^{mT} f_{1x_1} \sin 2\sigma_1 t\ dt + \sigma_1 \int_0^{mT} f_{1x_1'} \cos 2\sigma_1 t\ dt\right]^2, \end{aligned} \tag{3.7}$$

and each of the functions are evaluated at $x_0(0, t)$, $x_0'(0, t)$, t.

Except for the condition $A^2 \neq B$ this generalizes a result of L. Mandelstam and N. Papalexi ([12]) for the case $n = 1$. This result was also obtained for

$n = 1$ by H. R. Bailey and R. A. Gambill ([2]) using the above method of successive approximations. Also, for arbitrary n and all of the conditions in (3.5) violated (i.e., all of the characteristic multipliers of the linear variational equation equal for $\epsilon = 0$), H. R. Bailey and R. A. Gambill ([2]) obtained explicitly a matrix of order $2n$ whose eigenvalues determine the stability properties of $x_0(\epsilon, t)$. These stability theorems could also probably be obtained from the recent results of I. I. Blehman ([3]) and J. A. Nohel ([18]).

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