A DISCUSSION OF DIFFERENTIAL EQUATIONS ON PRODUCT SPACES

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Introduction

This report is based on the results of a joint paper by Jack K. Hale and Arnold Stokes entitled "Behavior of Solutions near Integral Manifolds", to be published in Archives for Rational Mechanics and Analysis. This paper will be referred to here as H. S. As this paper is extensive, with a number of theorems and applications, a detailed summary would probably be so lengthy and so detailed as to be unreadable.

Thus, the authors felt that it would be more useful to discuss in general terms the type of problem considered in the paper and relate the results to those of other authors. As this problem leads to considering systems which split into sub-systems, each sub-system possessing different properties, we include in the last portion of the paper some observations on methods of discussing such systems. Such equations have been referred to by Lefschetz ([3], p. 117) as "product space equations" and have been considered by Persidskii ([6]), Malkin ([5]), Dyhman ([2a]) and others.

Throughout this article, all functions are assumed Lipschitzian, defined in suitable domains, etc. As the emphasis is on basic concepts, unnecessary detail has been suppressed throughout; it is hoped not to the point of obscurity. As notation, R^k is k-dimensional Euclidean space; if $x \in R^k$, then $\| x \|$ is any norm in R^k; C^k is the class of continuous functions with continuous derivatives up through order k.

Section 1

Consider the equation

$$\dot{x} = Ax + f(t, x), \qquad x \text{ an } n\text{-vector}, \tag{1}$$

where $f(t, x) = o(\| x \|)$ as $\| x \| \to 0$, uniformly in t, and A is a constant matrix, all of those eigen-values have negative real parts. Then the origin is asymptotically stable, and solutions beginning near the origin tend to 0 exponentially. These results, while important, have long been known; see Lyapunov [4].

For many reasons, one would like to consider how the behavior of (1) is altered if the system is disturbed. In this case, there is the result, due to Malkin ([5], p. 303), (see also Coddington and Levinson [1], Remark, p. 328) that in the equation

$$\dot{x} = Ax + f(t, x) + g(t, x), \tag{2}$$

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where A and f are as in (1), and $\| g(t, x) \| \leq \eta$, for $t \geq 0$, $\| x \|$ bounded, the origin is stable if η is sufficiently small. Here observe that the stability of the origin has been preserved, (even though the origin may not be a solution of (2)), but the presence of g destroys the asymptotic stability. The function g has an obvious physical interpretation in this case.

If, however, one requires that $\| g \| \leq \eta$, and $g \to 0$ as $t \to \infty$, for $\| x \|$ bounded, the origin in (2) is asymptotically stable, if η is small enough (see, e.g., Coddington and Levinson [1, p. 327]), although no longer do the solutions approach zero exponentially.

In fact, if $x(t)$ is a solution, $x(t) \to 0$ as $t \to \infty$ at very nearly the same rate as g does; for instance, if $\int_0^\infty \| g(t) \| \, dt < \infty$, $\int_0^\infty \| x(t) \| \, dt < \infty$, etc. (see H.S.)

Note that if $g \to 0$ for $\| x \|$ small, as $t \to \infty$, one may assume $\| g \|$ small for all $t \geq 0$, as does Malkin, or discuss the equation for $t \geq T$, where T is such that $\| g(t, x) \| \leq \eta$ for $t \geq T$. This latter course was chosen by Coddington and Levinson, and the authors.

Section 2

In every equation considered in Section 1 all the components of a solution behaved similarly; that is, all components went to zero, or stayed small. But many circumstances arise wherein it is either not natural or not possible to require that all components have the same behavior. In (1), for instance, if some of the eigen-values had zero real parts, with the remaining real parts negative, it would be natural to divide the system into two parts. Let us consider some examples in which one or more zero eigen-values occur in the linear term.

In the equation

$$\dot{x} = X(x), \qquad x \text{ an } n\text{-vector}, \ X \in C^2, \tag{3}$$

assume that there exists a periodic solution $x_0 = x_0(\omega t)$, where $x_0(2\pi + s) \equiv x_0(s)$ for all s. Assume further that, in the variational equation

$$\dot{y} = \frac{\partial X(x_0(\omega t))}{\partial x} y, \tag{4}$$

$(n - 1)$ of the characteristic exponents have negative real parts. Then, in the x-space, the limit cycle is asymptotically stable. However, more is true also.

As (3) is autonomous, $x_0(\omega t + \varphi)$ is also a solution, for $\varphi \in R$. Thus in the (x, t) space, $x_0(\omega t + \varphi)$ generates a cylinder C as t and φ vary. Then, under the same assumptions, this cylinder is not only asymptotically stable, that is, nearby solutions approach the cylinder (in this case exponentially), but also, if $x(t)$ is a solution of (3) sufficiently near C, then there exists φ_0 such that $x(t) - x_0(\omega t + \varphi_0) \to 0$ as $t \to \infty$, that is, x approaches a particular periodic solution lying on the cylinder (see Coddington and Levinson [1], p. 323).

Now it is possible to introduce local coordinates around the cylinder C (see, e.g., Lefschetz [3], p. 149, or H. S.), and in these coordinates, (3) has the form:

$$\begin{aligned} \dot{s} &= \omega + \Phi(t, s, h), \\ \dot{h} &= Hh + H_1(t, s, h), \end{aligned} \tag{5}$$

where s is a scalar, h is an $(n - 1)$-vector, the eigen-values of H are the non-zero characteristic exponents of (4), and Φ and H_1 are second order in h uniformly in t and s. The cylinder C is given by $h = 0$.

Then, if $x(t)$ is a solution of (3), $x(t) \to C$ is equivalent to $h(t) \to 0$ as $t \to \infty$, $s(t)$ defined for all $t \geq 0$; and $x(t) - x_0(\omega t + \varphi_0) \to 0$ for some φ_0, is equivalent to $h(t) \to 0$, $s(t) - (\omega t + \varphi_0) \to 0$ as $t \to \infty$. Clearly, it is natural to divide this system into two parts, and ask different questions about the two sub-systems.

As was done in Section 1, consider (3) in the presence of a disturbance, that is, the equation

$$\dot{x} = X(x) + X^*(t, x), \tag{6}$$

where X is given in (3), and $\| X^*(t, x)\| \leq \eta$ for x in some bounded region containing x_0. As may be anticipated from the discussion in Section I, and the form of equation (5), for η sufficiently small, C is stable but not asymptotically stable.

If $X^* \to 0$ as $t \to \infty$, then, as before in Section 1, C is now asymptotically stable, though not exponentially (see H.S.). And, if for x bounded,

$$\int_0^\infty \| X^*(t, x)\| \, dt < \infty,$$

and if $x(t)$ is a solution to (6) sufficiently near C, then there exists a φ_0 such that $x(t) - x_0(\omega t + \varphi_0) \to 0$ as $t \to \infty$ (or, in local coordinates, $h(t) \to 0$, $s(t) - (\omega t + \varphi_0) \to 0$ as $t \to \infty$) (see H.S.). In this last case, $\int_0^\infty \| X^*(t, x)\| \, dt < \infty$ implies $\int_0^\infty \| h(t)\| \, dt < \infty$, as in Section 1.

Section 3

For another, more interesting example of a system which naturally splits into sub-systems, consider

$$\dot{x} = X(x), \qquad x \text{ an } n\text{-vector}, \quad X \in C^2, \tag{7}$$

and now assume that there exists a $(k + 1)$-parameter family of periodic solutions $x_0 \equiv x_0(\omega(b)t + \varphi, b)$, where $\varphi \in R$, $b = (b_1, \cdots, b_k) \in V \subset R^k$, V an open set, $\omega(b) > 0$ for $b \in V$, and $x_0(s, b) \equiv x_0(s + 2\pi, b)$, for all s, and all $b \in V$. We assume throughout this section that $k \geq 1$, for otherwise, the discussion in Section 2 applies.

Now assume that the linear variational equation

$$\dot{y} = \left(\frac{\partial X(x_0(\omega(b)t + \varphi, b))}{\partial x}\right) y \tag{8}$$

has $n - (k + 1)$ characteristic exponents with negative real parts for all $b \in V$. It is reasonable to expect then that the family of functions $\{x_0\}$ is asymptotically stable in some sense.

Let M be the manifold in (x, t)-space defined by the family of functions $\{x_0\}$ and let C_{b_0} be the cylinder defined by the family x_0 with $b = b_0$. Then M consists of the k-parameter family of cylinders C_{b_0}, $b_0 \in V$. Under the above assumptions, M is asymptotically stable (with an exponential approach), and, in addition, if $x(t)$ is a solution of (7) sufficiently near M, then there exist $b' \in V$, $\varphi_0 \in R$ such that $x(t) - x_0(\omega(b')t + \varphi_0, b') \to 0$ as $t \to \infty$, (see H. S.).

The method used for the proof of this result is to introduce local coordinates (s, a, h) in the neighborhood of M so that any particular solution on M is given by $s = \omega(a_0)t + \varphi_0$, $a = a_0$, $h = 0$, φ_0, a_0 constant.

To be more precise, choose $b_0 \in V$, let $a = b + b_0$, and now introduce local coordinates* around x_0 for $a \in S$, a bounded open sphere with center at the origin. In these coordinates, (7) has the form

$$\begin{aligned} \dot{s} &= \omega(a) + \Phi(t, s, a, h) \\ \dot{a} &= A(t, s, a, h) \\ \dot{h} &= H(a)h + H_1(t, s, a, h) \end{aligned} \tag{9}$$

where φ is a scalar, a is a k-vector, h is an n-$(k + 1)$ vector;

$$\| A(t, s, a, h) \| \leq L \| h \|,$$

for $a \in S$, $\| h \|$ small, and all t, s; Φ and H_1 are second order in $\| h \|$ uniformly in t, s (a is bounded), and all the eigen-values of $H(a)$ have negative real parts for $a \in S$.

The above stability result can now be stated in terms of the local coordinates in (9). If $h(0)$, $a(0)$, $s(0)$ are given and $\| h(0) \|$ is sufficiently small, then there exist a_0, φ_0 such that the solution of (9) through $h(0)$, $a(0)$, $s(0)$ satisfies the property that $h(t) \to 0$, $a(t) \to a_0$ and $s - \omega(a_0)t + \varphi_0 \to 0$, as $t \to \infty$. Furthermore $\| a(0) - a_0 \| \to 0$ as $\| h(0) \| \to 0$. With s absent, and A, H, and H_1 analytic in a, h, this theorem appears in Malkin [5], Dyhman [2a].

This is an example where a variety of behaviors is expected of the different components of a solution. In fact, equation (9), and in particular the last two equations in (9), and similar equations, have been studied at length. Before discussing these equations further, let us consider the effect on (7) of introducing disturbances. So consider

$$\dot{x} = X(x) + X^*(t, x), \tag{10}$$

with X as in (7), and now X^* must not only be small, but $X^* \to 0$ as $t \to \infty$ rather quickly; in fact, for x bounded, the authors require

$$\int_0^\infty \| X^*(t, x) \| \, dt < \infty.$$

* These coordinates exist if the matrix $[\partial x_0(s, b)/\partial s, \ \partial x_0(s, b)/\partial b]$ has rank $k + 1$ (see H. S.).

The reason for this is clear, if one recalls that the coordinates in (9) are valid only for $a \in S$. Thus, if these coordinates are to be used, the a-component of a solution must be bounded. But clearly a will be bounded in general only if

$$\int_0^\infty \| h(t) \| \, dt < \infty,$$

and from Section I, this requires the above conditions on X^*. Again note that here, and in Section II, rather than require $\| X^* \|$ small for $t \geq 0$, since $X^* \to 0$ as $t \to \infty$, the theorem can be stated for x bounded and $t \geq T$ instead, where $\| X^*(t, x) \|$ is small for $t \geq T$.

Now if $\int_0^\infty \| X^*(t, x) \| \, dt < \infty$, then (10), in local coordinates as in (9), behaves as follows: if $(h(t), a(t), s(t))$ is a solution of (10) sufficiently near M, that is, $\| h(0) \|$ sufficiently small, then $h(t) \to 0$, $a(t) \to a_0 \in S$, as $t \to \infty$, and $s(t)$ is defined for $t \geq 0$. This result implies that the solution not only approaches M but also approaches one of the cylinders C_{b_0} mentioned above. However, the solutions do not necessarily approach one of the functions x_0 on this cylinder. To obtain this stronger type of stability the authors require the stronger condition $\int_0^\infty \int_t^\infty \| X^*(u, x) \| \, du \, dt < \infty$, and then, in addition to the above, $s(t) - (\omega(a_0)t + \varphi') \to 0$ as $t \to \infty$, for some φ'.

The discussion above has been for the variable φ a scalar. In H. S., φ is a vector and the same type of results are valid. This case has many interesting applications; for example, one can consider perturbations of stable torii. For a discussion of this and numerous examples illustrating the theory, see H. S.

Without making further assumptions on the form of the function A, in (9), it seems that no results are obtainable in the cases $\| X^* \| \leq \eta$, or $\| X^* \| \leq \eta$ and $X^* \to 0$ as $t \to \infty$.

One possibility not investigated by the authors is the behavior of (9) under the assumption that there exists a Liapunov function for the last equation of (9) which implies that $h \to 0$ as $t \to \infty$, for s and a suitably restricted. If a is not present, as in Section 2, the existence of such a function trivially implies that the cylinder C is stable. However, to obtain the stronger results in Section 2, or to consider (9) in Section 3, one needs $\int_0^\infty \| h(t) \| \, dt < \infty$, where h is a part of a solution of (5) or (9). No general conditions on a Liapunov function which imply that h behaves in this manner are known to the authors.

Section 4

Consider the system of equations

$$(11) \qquad \begin{aligned} &\text{(a)} \quad \dot{y} = Y(y, z, t) \\ &\text{(b)} \quad \dot{z} = Z(y, z, t), \end{aligned}$$

where $y \in R^q$, $z \in R^r$, $Y(0, 0, t) = Z(0, 0, t) = 0$, and Y, Z are continuous and Lipschitzian in some neighborhood of the origin in $R^q \times R^r$. One now wishes to determine the behavior of (11) by examining separately (11a) and (11b).

Persidskii proceeds as follows (see Lefschetz [3], p. 118). Let $p(t)$ be a continuous r-vector, and consider the equation

$$\dot{y} = Y(y, p(t), t). \tag{12}$$

According to Persidskii, $y = 0$ is quasi-stable (quasi-unstable) whenever the following condition is (is not) fulfilled: Given $\epsilon > 0$, and $t_0 \geq 0$, there exists an $\eta(\epsilon)$, $0 < \eta(\epsilon) < \epsilon$, such that if $\| p(t_0) \| \leq \eta$, $\| p(t) \| \leq \epsilon$, $t \geq t_0$, then any solution $y(t)$ of (12), with $\| y(t_0) \| \leq \eta$ satisfies $\| y(t) \| < \epsilon$, $t \geq t_0$. System (11a) is said to be quasi-asymptotically stable if it is quasi-stable and whenever p is such that $p(t) \to 0$ as $t \to \infty$, then the solution of (12) also approaches zero as $t \to \infty$. By symmetry, the same notions are defined for (11b). Persidskii then states that if the origin is quasi-stable (quasi-asymptotically stable) for both (11a) and (11b), then the solution $y = 0$, $z = 0$, of (11) is stable (asymptotically stable) in the sense of Lyapunov. Also, if the origin is quasi-unstable for one of the two partial systems (11a) or (11b), then the solution $y = 0$, $z = 0$ of (11) is unstable.

The theorem on instability is not correct. As a counter-example consider the equation

$$\dot{a} = a^2 h \qquad \dot{h} = -h. \tag{13}$$

This is a special case of (9) without the variable s, and so there exists a neighborhood U of (0,0) such that if $(a(0), h(0)) \in U$, where $(a(t), h(t))$ is a solution of (13), then $h(t) \to 0$, $a(t) \to a_0$, as $t \to \infty$, and $\| a(0) - a_0 \|$ may be made as small as desired. Therefore, the solution $a = 0$, $h = 0$ is stable. But the solution $a = 0$ of the equation $\dot{a} = a^2 h$ is quasi-unstable since every nonzero solution of $\dot{a} = a^2 \epsilon$ with positive initial conditions approaches ∞ in a finite time for every $\epsilon > 0$.

This example points out how difficult it is to determine conditions for the instability of a system of equations by examining a partial system. For some results along this line, see the paper of M. R. Dyhman ([2]) (also S. Lefschetz [3], p. 134).

The theorem that quasi-asymptotic stability of (11a), (11b) implies quasi-asymptotic stability of (11) is also not true. For consider the equations, $\dot{x} = -x + y$, $\dot{y} = -y + x$. Then $x(t) + y(t) = x(0) + y(0)$ for every solution $x(t), y(t)$, and, therefore, this system is not asymptotically stable; but each subsystem is quasi-asymptotically stable.

An improvement on this method of approach to a system such as (11) can be obtained as follows: Introduce the function spaces E^q and E^r, where E^n is the space of all continuous functions from $[0, \infty)$ into R^n. It will not be necessary to consider these as topological spaces, just as sets of functions.

Define two sets $\mathcal{Y} = \{y \in E^q \mid \; \| y(t) \| < g_1(t), \text{ for all } t \geq 0\}$ and $\mathcal{Z} = \{z \in E^r \mid \; \| z(t) \| < g_2(t), \text{ for all } t \geq 0\}$, where $g_1, g_2 \in E^1$ are given functions that are positive for all $t \geq 0$. Let $\bar{\mathcal{Y}}, \bar{\mathcal{Z}}$ be similarly defined, with "$<$" replaced by "$\leq$". Then we have the following theorem.

THEOREM. Suppose g_1, g_2 and η are such that $\eta < g_1(0)$, $\eta < g_2(0)$, and further, in the equation

$$\dot{y} = Y(y, p(t), t), \tag{14}$$

if $\| y(0) \| \leq \eta$, and $p \in \bar{\mathcal{Z}}$, then the solution $y \in \mathcal{Y}$, and in

$$\dot{z} = Z(q(t), z, t) \tag{15}$$

if $\| z(0) \| \leq \eta$, and $q \in \bar{\mathcal{Y}}$, then the solution $z \in \mathcal{Z}$. Then if $\| y(0) \|$, $\| z(0) \| \leq \eta$, the solution (y, z) of (11) lies in $\mathcal{Y} \times \mathcal{Z}$.

Proof. The argument, due to Perron, is the same as that given by Persidskii. If $(y, z) \notin \mathcal{Y} \times \mathcal{Z}$, then there exists a first t_1 such that either $\| y(t_1 \| = g_1(t_1)$, or $\| z(t_1) \| = g_2(t_1)$. Note that $t_1 > 0$ since $\eta < g_1(0), g_2(0)$. Suppose

$$\| z(t_1) \| = g_2(t_1).$$

Then let $z^* \in \bar{\mathcal{Z}}$ be such that $z^*(t) = z(t)$ on $[0, t_1]$. Let y^* be the solution of (14) with $y^*(0) = y(0)$, $p(t) = z^*(t)$. Then $y^* \in \mathcal{Y} \subset \bar{\mathcal{Y}}$, so if $\bar{z}$ is the solution of (15) with $q(t) = y^*(t)$, $\bar{z}(0) = z(0)$, then $\bar{z} \in \mathcal{Z}$. But by the uniqueness of the solution of (11), $z^* = z$ on $[0, t_1]$ implies $y^* = y$ on $[0, t_1]$, which in turn implies $\bar{z} = z$ on $[0, t_1]$. But $\| z(t_1) \| = \| \bar{z}(t_1) \| < g_2(t_1)$ gives a contradiction, so that $(y, z) \in \mathcal{Y} \times \mathcal{Z}$.

It is clear that this argument extends trivially to n systems of the form $\dot{x}_i = X_i(x_1, \cdots, x_n, t)$, with n classes $\mathfrak{X}_i$, $\bar{\mathfrak{X}}_i$ defined, using n functions f_i, etc. Further, uniqueness is used only to clarify the argument, and could be dispensed with by a trivial restatement of the theorem.

As is clear, this alternative approach involves no new concepts, but it does provide greater clarity, as the difficulty involved in the definition of quasi-asymptotic stability is avoided, and more importantly, greater flexibility is obtained. It is clear that if $g_1(t) \equiv g_2(t) \equiv \epsilon$, then Persidskii's theorem on quasi-stability implying stability is obtained. But in addition, if $g_1(t) = \epsilon$, and $g_2(t) \to 0$ as $t \to \infty$, at a suitable rate, then systems such as (13) can also be examined.

In addition, sets of the type $\mathcal{Y}$ or $\mathcal{Z}$ can be used in the investigation of such diverse questions as existence in the large, boundedness or uniform boundedness, and ultimate boundedness (in the sense of Yoshizawa [8]). For illustration of such applications to a single equation, see Stokes [7]. The above argument shows clearly that if the solutions of the entire system are unique, then the method can be applied to show the presence of any of the properties above for each partial system.

For example, consider the results of the authors regarding the behavior of (9). In fact, for (9) as written, and $\epsilon > 0$ the sets $\mathcal{S} = \{s \in E^1 \mid \| s(t) \| < g(t)\}$, where $g \in E^1$, $g > 0$, $\mathcal{A} = \{a \in E^k \mid \| a(t) - a_0) \| < m(t)\}$, where $m \in E^1$, $0 < m(t) < K\epsilon$, where K is a constant, and a_0 is the initial condition for a, and $\mathcal{H} = \{h \in E^{n-(k+1)} \mid \| h(t) \| < \epsilon e^{-\sigma t}\}$, where $\sigma > 0$ is related to the matrix $H(a_0)$, were used to provide some of the desired results. The conclusion that the solution $(s, a, h) \in \mathcal{S} \times \mathcal{A} \times \mathcal{H}$ then implies that s exists in the large, a is bounded, and $h(t) \to 0$ as $t \to \infty$. Of course, this is under the restriction that $\| h(0) \| \leq \eta < \epsilon$, for a suitable η.

To obtain the fact that $\lim_{t\to\infty} a(t) = a'$ exists, one can use the fact that as a is a solution, for $t_2 > t_1 > 0$,

$$\| a(t_2) - a(t_1) \| = \| \int_{t_2}^{t_1} \dot{a}(u)\, du \| \leq L \int_{t_1}^{t_2} \| h(u) \|\, du \leq \epsilon L \int_{t_1}^{t_2} e^{-\sigma u}\, du.$$

A similar argument shows that $s(t) - (\omega(a')t + s') \to 0$ as $t \to \infty$.

In the paper of the authors, the matrix $H(a)$ is replaced by $H(a, s, t)$, and in one theorem the principal matrix solution of $\dot{h} = H(a, l(t), t)h$ is assumed to be properly behaved only for those l such that $\dot{l} = \omega(a')$ for some a'. This requires a slightly more subtle argument as one now considers an equation of the form $\dot{h} = H(a', l(t), t)h + (H(a, s, t) - H(a', l(t), t))h + H_1(a, h, s, t)$. Then the second term is bounded by $L' \| h \| \cdot [\| a - a' \| + \| s - l(t) \|]$, and it is necessary to produce an argument to show that $\| s(t) - l(t) \|$ is small, if s is a solution, where as a further complication, $l(t)$ involves the $\lim_{t\to\infty} a(t)$, where a is a solution.

But in some instances, the straightforward approach works quite easily. For other examples of systems with two or three sub-systems, to which this method applies directly, see Dyhman ([2b]).

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