# INVARIANT MANIFOLDS[1]

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Manifolds defined by solutions to ordinary differential equations were studied by Krylov and Bogoliubov [1] in 1934 and in the last decade by many Russian and American mathematicians. The paper by Jack Hale [2] contains a discussion of much of this research. In 1955, John McCarthy [3] wrote a paper, "The Stability of Invariant Manifolds," in which transformations of the type generated by the differential equation problems were used in a more general setting. He considered homeomorphisms $T_\lambda$ (depending differentiably on a parameter $\lambda$) of a Riemannian manifold $W$ onto itself. Sufficient conditions were given for the existence of neighboring submanifolds $V_\lambda$ which were invariant under $T_\lambda$.

The manifold $W$ can be, for example, the two dimensional phase space of a perturbed van der Pol's equation

$$x'' + a(x^2 - 1)x' + x = \lambda \sin t.$$

Let the mapping $T_\lambda$ be induced by the point mapping

$$x(0, \lambda) \rightarrow x(2\pi, \lambda), \qquad x'(0, \lambda) \rightarrow x'(2\pi, \lambda).$$

The limit cycle $V_0$ is invariant under $T_0$. It follows from a theorem of N. Levinson [4] that there exist nearby curves $V_\lambda$ which are invariant under $T_\lambda$.

Levinson's hypothesis, which required that the unperturbed $n$th order differential equation have $n - 1$ characteristic exponents with negative real parts, has been weakened to the requirement that the characteristic exponents have non-zero real parts (see [5], [6], and [7]). As McCarthy pointed out, his result contains Levinson's, but not its conditionally stable generalization. It is the purpose of this paper to report that McCarthy's result can be extended to cover the conditionally stable case. The new result has been applied to a differential equation which, because of critical points, is not covered by the extensions of Levinson's theorem given in [5], [6], [7].

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Bibliography

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