DYNAMIC PROGRAMMING AND CLASSICAL ANALYSIS

BY RICHARD BELLMAN

1. Introduction

Over the last ten years, research in the field of dynamic programming has assumed many different forms. Sometimes, the emphasis has been upon questions of formulation in analytic terms and concepts ([1, 2]); sometimes upon the problems of existence and uniqueness of solutions of the functional equations derived from the underlying processes ([2], [3], [4]); occasionally upon the actual analytic structure of the solutions of these equations ([5], [6], [7], [8], [9]); sometimes upon the computational aspects ([10], [11]); and sometimes upon the applications to control processes ([2], [12], [13]), to trajectories of various types ([13], [14]), to operations research ([15], [16], [17]), to mathematical economics ([18], [19]).

Inevitably, the result of this quasi-ergodic behavior has been to ignore a number of significant problems, and to treat a number of others in cavalier fashion. In this exposition, we wish to focus attention upon a number of interesting, difficult, and significant questions in analysis which arise naturally out of the functional equation technique of dynamic programming. Our aim is to show that this theory constitutes a natural extension of classical investigations and that the corresponding problems are natural generalizations of problems of classical analysis.

As always in studying new areas, there is the hope that light thrown in this virgin territory will be reflected back upon still hidden parts of the classical domain.

We shall restrict our attention here to deterministic processes, reserving for other times any discussion of more complex and arcane processes arising from the study of stochastic and adaptive processes ([2]). These new areas of analysis where many different sub-disciplines such as algebra, topology, differential equations and probability theory merge and lose their separate identity in all-embracing problems offer bountiful and boundless regions for research. The reader familiar with the concepts and problems discussed here can readily construct for himself corresponding questions involving uncertainty.

2. Multistage Decision Processes

Dynamic programming is a mathematical theory of multistage decision processes. Expositions of this theory may be found in [1] and [2]. The problems posed below will be very much more meaningful within the context of dynamic programming. The reader who wishes, however, may ignore all questions of motivation and regard the problems that follow as conundrums pulled out of the blue.

3. The Calculus of Variations

Perhaps the most interesting and important example of a multistage decision process of continuous type is the calculus of variations. Consider the scalar variational problem of maximizing the integral

$$J(y) = \int_0^T F(x, y)\, dt, \tag{1}$$

over all functions y, where x and y are related by means of the differential equation

$$\frac{dx}{dt} = G(x, y), \qquad x(0) = c. \tag{2}$$

Introducing the function

$$f(c, T) = \operatorname*{Max}_y J(y), \tag{3}$$

we obtain from the principle of optimality ([1]), the nonlinear partial differential equation

$$\frac{\partial f}{\partial T} = \operatorname*{Max}_v \left[F(c, v) + G(c, v) \frac{\partial f}{\partial c} \right], \tag{4}$$

with the initial condition

$$f(c, 0) = 0. \tag{5}$$

See [1], [2].

We shall use this equation as a pivot for some of the subsequent problems we shall present.

4. Existence and Uniqueness

The equation in (3.4) is derived in a formal fashion, much as the Euler equation is customarily obtained. The first problem we pose is that of finding conditions upon the functions $F(x, y)$ and $G(x, y)$ which ensure the existence and uniqueness of a solution of the unconventional nonlinear partial differential equation of (3.4).

One way to approach this problem is to employ the results of the classical calculus of variations. These enable us to establish the existence of a solution of the original variational problem, under certain rather rigid conditions upon $F(x, y)$ and $G(x, y)$.

This is not a particularly satisfactory procedure for a number of reasons. In the first place, we would like to use Equation (3.4) to resolve the original variational problem. Secondly, we want to use analogues of (3.4) to study variational problems which partially or wholly escape the classical theory. This will be the substance of the following section.

5. Constraints

A question of much analytic interest, and one which in any case is rudely thrust upon us by a great number of feedback control and trajectory processes, is that of solving the variational problem of (3.1) and (3.2) under the additional constraint

$$|y(t)| \leq k, \qquad 0 \leq t \leq T. \tag{1}$$

It is certainly surprising that a simple condition of this nature should so effectively stymie the usual approach, but, nonetheless, true. For a detailed discussion, see [2], [20].

As a result of the presence of the foregoing restriction, the equation of (3.4) is replaced by the equation

$$\frac{\partial f}{\partial T} = \operatorname*{Max}_{|v| \leq k} \left[F(c, v) + G(c, v) \frac{\partial f}{\partial c} \right]. \tag{2}$$

One might surmise that the presence of the constraint would simplify the investigation of existence and uniqueness of solution, as it certainly does the computational solution of equations of this nature. However, equations of the foregoing type have not been rigorously investigated as yet.

6. Discrete Version—I

A standard route to the goal of existence of a solution of a functional equation involving derivatives is by way of difference equations. This approach is of the utmost importance in connection with the numerical solution of these equations.

Let us then in place of (3.4) write the recurrence relation

$$\begin{aligned} &\frac{f(c, T + \Delta) - f(c, T - \Delta)}{2\Delta} \\ &\qquad = \operatorname*{Max}_{v} \left[F(c, v) + G(c, v) \left(\frac{f(c + \delta, T) - f(c - \delta, T)}{2\delta} \right) \right], \end{aligned} \tag{1}$$

where c and T assume respectively values of the form $\pm k\delta$, $l\Delta$. As $\Delta \rightarrow 0$, $\delta \rightarrow 0$, this relation formally approaches that of (3.4).

The problem now is that of determining the connection between the solution of (1) and the possible solution of (3.4), first under the assumption that (3.4) has a solution and secondly in the hope of using the solution of (1) to establish the existence of a solution of (3.4).

Closely associated with questions of this nature is the question of numerical stability of a computational algorithm such as (1). Finally, let us note that this is only one of a large set of possible discrete approximations to (3.4).

7. Discrete Version—II

To obtain a discrete version of a different type, we use an idea of some importance as far as approximation techniques are concerned. In place of approximating to the exact equation describing the process, we can employ the exact equation for an approximating process.

Hence, in place of the continuous decision process described by the equations of (4.1) and (4.2), let us consider the problem of determining the sequence $\{y_k\}$ which minimizes the function

$$J_N(y) = \sum_{k=0}^{n-1} F(x_k, y_k)\Delta, \tag{1}$$

where x_k and y_k are related by means of the equation

$$x_{k+1} - x_k = G(x_k, y_k)\Delta, \qquad x_0 = c, k = 0, 1, 2, \cdots. \tag{2}$$

Here $x_k = x(k\Delta)$, $y_k = y(k\Delta)$, and $N\Delta = T$.

Setting

$$f_N(c) = \underset{y}{\text{Minl}}\, J_N(y), \tag{3}$$

we obtain the recurrence relation

$$f_N(c) = \underset{v}{\text{Max}}\,[F(c, v)\Delta + f_{N-1}(c + G(c, v)\Delta)], \tag{4}$$

with

$$f_0(c) = \underset{v}{\text{Max}}\, F(c, v)\Delta. \tag{5}$$

Recurrence relations of this nature have been quite successful in the obtaining of computational solutions; see [10, 14, 21].

Observe that as $\Delta \to 0$, the equation in (4) formally reduces to that of (3.4). Yet, unlike (6.1), it involves only one small quantity Δ, and no ratio of small quantities such as Δ/δ. This is a most important point in connection with numerical stability.

A small amount of work has been done on the subject of the convergence of the solution of (4) to the solution of (3.4); see [22], [1], [23], [24]. Much remains to be done.

8. An Application

We have mentioned above the possibility of applying ideas and techniques developed in the new field of multistage decision processes to classical equations. Let us give an example of this.

Consider the partial differential equation

$$u_t = uu_x, \qquad u(x, 0) = g(x), \tag{1}$$

an equation which possesses a shock discontinuity. In place of the usual finite difference scheme, à la (6.1), let us borrow the approach of (7.4) and use the relation

$$u(x, t + \Delta) = u(u + u(x, t)\Delta, t), \tag{2}$$

as an approximation to (1).

Excellent results have been obtained, even in the immediate neighborhood of the shock, using this algorithm; see [25]. No work has been done, however, on the question of the convergence of the solution of (2) to the solution of (1) as $\Delta \to 0$, for this equation or for more general equations. For another application of this idea, see [26].

9. Approximation in Policy Space

Another attack on the basic problem of establishing existence and uniqueness of solutions of the equation in (4.7) or (6.2) is by means of the method of successive approximations. In place of the usual approach, let us invoke the technique of *approximation in policy space*; see [1], [2].

We begin by guessing an initial *policy*, $v_0 = v_0(c, T)$, and using this policy to determine a return function $f_0 = f_0(c, T)$, by means of the linear partial differential equation

$$(1) \qquad \frac{\partial f_0}{\partial T} = F(c, v_0) + G(c, v_0) \frac{\partial f_0}{\partial c}, \qquad f_0(c, 0) = 0.$$

Having determined $f_0(c, T)$, let us determine the function v_1 by the condition that it maximize the function

$$(2) \qquad F(c, v) + G(c, v) \frac{\partial f_0}{\partial c}.$$

Using this new policy-function, let us determine the new return function f_1 by way of the linear partial differential equation

$$(3) \qquad \frac{\partial f_1}{\partial T} = F(c, v_1) + G(c, v_1) \frac{\partial f_1}{\partial c}, \qquad f_1(c, 0) = 0.$$

Continuing in this fashion, we obtain a sequence of functions $\{f_k\}$ and $\{v_k\}$. In view of the manner in which v_1 is determined, it is clear that

$$(4) \qquad \begin{aligned} \frac{\partial f_0}{\partial T} &= F(c, v_0) + G(c, v_0) \frac{\partial f_0}{\partial c}, \\ &\le F(c, v_1) + G(c, v_1) \frac{\partial f_0}{\partial c}. \end{aligned}$$

From this, we would expect to find that $f_0(c, T) \le f_1(c, T)$ for $T \ge 0$, and this is actually the case. Generally, the merit of this approximation method is that it yields monotonic behavior,

$$(5) \qquad f_0(c, T) \le f_1(c, T) \le \cdots \le f_n(c, T) \le \cdots.$$

The important problem is to determine conditions under which this sequence converges to a solution of (3.4), and, of course, to investigate the application of this technique to more general equations. Some preliminary results for a much simpler problem may be found in [1], Chapter 11.

10. Positive Operators

A crucial result in the foregoing procedure is the conclusion $f_1 \geq f_0$ for $T \geq 0$ as a consequence of the equality

$$\frac{\partial f_1}{\partial T} = F(c, v_1) + G(c, v_1) \frac{\partial f_1}{\partial c}. \tag{1}$$

and the inequality

$$\frac{\partial f_0}{\partial T} \leq F(c, v_1) + G(c, v_1) \frac{\partial f_0}{\partial c}, \tag{2}$$

granted common initial conditions at $T = 0$.

Questions of this nature are part of the theory of positive operators. Given a relation $Au \geq 0$, where A is an operator, we wish to determine when this implies $u \geq 0$. This type of investigation was initiated by Caplygin ([27]); see [28], [29], [30] for further results and references. Again much remains to be done in this field.

11. Quasilinearization

Let us now indicate another application of new techniques developed in the theory of dynamic programming to classical analysis. We have just seen, in §9, that equations of the special form of (3.4) can be approached along a route which is not open to the equations of classical analysis. In view of the monotonicity of approximation, a very valuable analytic and computational aid, it may be worth devoting some effort to the question of converting an equation of conventional type into one of the form of (3.4).

What we are doing is transforming an equation arising from a *descriptive* process into one which arises from a *variational* process. This, of course, is a familiar idea in analysis and one of great power and versatility. The way in which we do it is, however, new.

To give a simple example of the technique which can be used, consider the Riccati equation

$$u' = u^2 + a(t), \qquad u(0) = c. \tag{1}$$

Write

$$u^2 = \operatorname*{Max}_v \,(2uv - v^2), \tag{2}$$

so that (1) assumes the form

$$u' = \operatorname*{Max}_v \,[2uv - v^2 + a(t)], \qquad u(0) = c. \tag{3}$$

Consider the related linear equation

$$U' = 2Uv - v^2 + a(t), \qquad U(0) = c, \tag{4}$$

whose solution we write in the form

$$U = T(v, t). \tag{5}$$

We suspect that

$$u = \operatorname*{Max}_{v} U = \operatorname*{Max}_{v} T(v, t), \tag{6}$$

a fact which is a consequence of the positivity of the operator $d/dt - 2v$, in the sense of §10. Since we can obtain an explicit representation for $T(v, t)$ in terms of integral equations, (6) furnishes an interesting representation for the solution of (1).

Generally, if $f(u, t)$ is convex as a function of u for $0 \leq t \leq t_0$, we can write

$$f(u, t) = \operatorname*{Max}_{v} [f(v, t) + (u - v)f_v(v, t)], \tag{7}$$

and if $f(u, t)$ is concave, we can write

$$f(u, t) = \operatorname*{Min}_{v} [f(v, t) + (u - v)f_v(v, t)]. \tag{8}$$

These representations enable us to treat differential equations of the form

$$u' = f(u, t), \tag{9}$$

and, more generally, to transform functional equations of the form

$$Au = f(u, t), \tag{10}$$

where A is an operator, into quasilinear equations of the form

$$Au = \operatorname*{Max}_{v} [f(v, t) + (u - v)f_v(v, t)]. \tag{11}$$

Having obtained this form, we can employ approximation in policy space, and proceed as above. This representation also has important computational advantages. For a more detailed discussion, with many examples, see [28], [29].

12. Semi-Group Theory

The modern theory of semi-groups of transformations ([31]), deals with functional equations of the form

$$\frac{\partial u}{\partial t} = Au, \tag{1}$$

where A is a linear operator. A particular equation of this type is the linear partial differential equation of (9.1). The functional equations associated with dynamic programming processes of continuous type have the form

$$\frac{\partial u}{\partial t} = \operatorname*{Max}_{v} [A(v)\, u + b(v)]. \tag{2}$$

In view of the fact that (2) contains (1), we can expect a diffusion of knowledge in both directions. In the first place, in view of the quasilinearity of (2), we may expect that a large part of semi-group theory can be applied to the question of the existence of solutions of (2). In this way we would hope to obtain far stronger results in the calculus of variations than those currently available. In particular, we would aspire to a more complete theory for variational problems with constraints.

Secondly, we may expect to utilize the results of the linear theory, and some of the results and techniques of dynamic programming, to obtain a theory of nonlinear equations which can be written in the form of (2).

13. Multidimensional Variational Problems

Consider the problem of maximizing the integral

$$J(u) = \int_R f(u, u_x, u_y)\, dA, \tag{1}$$

where the integration is over the interior of a two-dimensional region R, and the value of u is prescribed on the boundary B of R,

$$u = g(P), \qquad P \in B. \tag{2}$$

In a procedure completely analogous to the one-dimensional case, we may introduce the functional

$$f(g(P); R) = \operatorname*{Max}_u J(u). \tag{3}$$

Considering a sequence of shrinking regions, it is not difficult to obtain a functional equation for $f(g(P); R)$. The derivatives that occur will now be functional derivatives; cf. [32], [33].

Although this technique has been used to obtain the Hadamard variational formula for the Green's function of a region, and other results ([33], [34], [35], [36]), no work has been done on the existence and uniqueness of solutions of equations of this nature.

An interesting side problem associated with variational questions of this nature is that of determining the functional form of $f(g(P); R)$. Even for the classical problem involving the Dirichlet functional,

$$J(u) = \int_R (u_x^2 + u_y^2)\, dA, \tag{4}$$

deep and subtle analysis is required; see Osborn ([37]), where further references may be found.

14. Stability and Asymptotic Behavior

As soon as the fundamental questions of existence and uniqueness of solution have been disposed of, we can turn to the deeper and more interesting problems of the analytic structure of the solution. In particular, we would like to examine the stability properties of the solution and its asymptotic behavior as $t \to \infty$. A small amount of work has been done in this direction, but no systematic theory has been constructed; see [17, 38, 39].

A theory of this nature can be based upon an extension of the present theory of positive operators; see [40, 41, 42].

15. Implicit Variational Problems

So far we have considered variational processes of fixed duration. It is of interest to consider processes in which the upper limit T depends upon the policy used, and those of still more general nature. These are examples of *implicit variational* problems.

One of the most important examples of this type of problem is furnished by "bang-bang" control. Suppose that we have a system S ruled by a vector differential equation

$$\frac{dx}{dt} = g(x, y), \qquad x(0) = c, \tag{1}$$

where the components of y are subject to the restrictions

$$\begin{aligned} &\text{(a)} \quad |y_i(t)| \leq k_i, \quad i = 1, 2, \cdots, N, \quad \text{or} \\ &\text{(b)} \quad y_i(t) = \pm k_i, \quad i = 1, 2, \cdots, N. \end{aligned} \tag{2}$$

We wish to choose y in such a way as to minimize the time required to force the system from the initial state c to another state, say 0.

For the linear case,

$$\frac{dx}{dt} = Ax + y, \qquad x(0) = c, \tag{3}$$

a great deal has been done using a variety of techniques; see [43], [44], [45] for further references.

16. Further Directions

It is easy to pose a number of additional questions of interest if we admit two person processes, multistage games (cf. [4], [46]); and if we introduce stochastic and adaptive processes in general ([1], [2], [39], [47]).

The range of investigation is now so broad, and so different in many ways from the classical tableau, that we feel it better to present discussions of this nature in separate publications.

Bibliography

[1] R. Bellman, Dynamic Programming, Princeton Univ. Press, Princeton, N. J., 1957.

[2] R. Bellman, Adaptive Control Processes, A Guided Tour, Princeton University Press, 1960.

[3] R. Bellman, *Some functional equations in the theory of dynamic programming—I: functions of points and point transformation*, Trans. Amer. Math. Soc., vol. 80 (1955), pp. 51–71.

[4] R. Bellman, *Functional equations in the theory of dynamic programming—III*, Rend. Cir. Mate. Palermo, ser. 2, tomo 5 (1956), pp. 1–23.

[5] R. Bellman, *A variational problem with constraints in dynamic programming*, J. Soc. Ind. Appl. Math., vol. 4 (1956), pp. 48–61.

[6] R. Bellman, *On a class of variational problems*, Q. Appl. Math., vol. 14 (1957), pp. 353–359.

[7] R. Bellman, *Functional equations in the theory of dynamic programming—V: positivity and quasi-linearity*, Proc. Nat. Acad. Sci. USA, vol. 41 (1955), pp. 743–746.

[8] R. Bellman, I. Glicksberg, and O. Gross, *On the optimal inventory equation*, Management Sci., vol. 2 (1955), pp. 83–104.

[9] R. Bellman, *On some applications of dynamic programming to matrix theory*, Illinois J. Math., vol. 1 (1957), pp. 297–301.

[10] R. Bellman and S. Dreyfus, Computational Aspects of Dynamic Programming, Princeton Univ. Press, Princeton, N. J., to appear.

[11] R. Bellman and S. Dreyfus, *A bottleneck situation involving interdependent industries*, Naval Research Log. Q., vol. 5 (1958), pp. 21–28.

[12] R. Bellman, S. Dreyfus, and R. Kalaba, *Dynamic programming trajectories and space travel*, Proc. Fourth Symp. on Astronautics, Los Angeles, 1959, to appear.

[13] R. Bellman and S. Dreyfus, *An application of dynamic programming to the determination of optimal satellite trajectories*, J. Brit. Interplanetary Soc., vol. 17, 1959–1960, pp. 78–83.

[14] T. Cartaino and S. Dreyfus, *Application of dynamic programming to the airplane minimum time-to-climb problem*, Aero. Eng. Rev. (1957).

[15] S. Dreyfus, *A note on an industrial replacement process*, Operational Research Q. (December, 1957).

[16] R. Bellman, *Equipment replacement policy*, J. Soc. Indust. Appl. Math., vol. 3 (1955), pp. 133–136.

[17] R. Howard, *Discrete Dynamic Programming Processes*, Thesis, Massachusetts Institute of Technology, 1958.

[18] R. Beckwith, *Analytic and Computational Aspects of Dynamic Programming Processes of High Dimension*, Ph.D. Thesis, Purdue University, 1959.

[19] R. Bellman, I. Glicksberg, and O. Gross, *On the optimal inventory equation*, Management Science, vol. 2 (1955), pp. 83–104.

[20] R. Bellman, *Dynamic programming and its application to the variational problems in mathematical economics*, Proc. Symp. in Calculus of Variations and Applications, Amer. Math. Soc., Chicago, April, 1956, pp. 115–138; published by McGraw-Hill Book Co., Inc., New York.

[21] R. Bellman and S. Dreyfus, *On the computational solution of dynamic programming processes—I: on a tactical air warfare model of Mengel*, J. Oper. Res., vol. 6 (1958), pp. 65–78.

[22] R. Bellman, *Functional equations in the theory of dynamic programming—VI: a direct convergence proof*, Ann. of Math., vol. 65 (1957), pp. 215–223.

[23] H. Osborn, *The problem of continuous programs*, Pacific J. Math., vol. 6 (1956), pp. 721–731.

[24] W. Fleming, *Discrete Approximations to Some Continuous Dynamic Programming Processes*, The RAND Corporation, Research Memorandum RM-1501, June 2, 1955.

[25] R. Bellman, I. Cherry, and G. M. Wing, *A note on the numerical integration of a class of nonlinear hyperbolic equations*, Q. Appl. Math., vol. 16 (1958), pp. 181–183.

[26] R. Bellman, *On the nonnegativity of solutions of the heat equation*, Bull. Unione Matematico, vol. 12 (1957), pp. 520–523.

[27] S. Caplygin, A New Method for the Approximate Integration of Solution of Differential Equations, Moscow-Leningrad, 1950.

[28] R. Bellman, *Functional equations in the theory of dynamic programming—V: positivity and quasi-linearity*, Proc. Nat. Acad. Sci. USA, vol. 41 (1955), pp. 743–746.

[29] R. Kalaba, *On nonlinear differential equations, the maximum operation and monotone convergence*, J. Math. and Mech., vol. 8 (1959), pp. 519–574.

[30] E. F. Beckenbach and R. Bellman, *Inequalities*, vol. 1, Ergebnisse der Math., to appear.

[31] E. Hille and R. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ., vol. 31, 1958.

[32] R. Bellman, *Dynamic programming and a new formalism in the theory of integral equations*, Proc. Nat. Acad. Sci. USA, vol. 41 (1955), pp. 31–34.

[33] R. Bellman and H. Osborn, *Dynamic programming and the variation of Green's functions*, J. Math. and Mech., vol. 7 (1958), pp. 81–86.

[34] R. Bellman, *Functional equations in the theory of dynamic programming—VIII: the variation of Green's functions for the one-dimensional case*, Proc. Nat. Acad. Sci. USA, vol. 43 (1957), pp. 839–841.

[35] R. Bellman and S. Lehman, *Functional equations in the theory of dynamic programming—IX: variational analysis, analytic continuation, and imbedding of operators*, Proc. Nat. Acad. Sci. USA, vol. 44 (1958), pp. 905–907.

[36] R. Bellman and S. Lehman, *Functional equations in the theory of dynamic programming—X: resolvents, characteristic functions and values*, Duke Math. J., vol 27, 1960, pp. 55–70.

[37] H. Osborn, *The Dirichlet functional*, J. Math. Analysis and Appl., vol. 1, 1960, pp. 61–112.

[38] R. Bellman, *A Markovian decision process*, J. Math. and Mech., vol. 6 (1957), pp. 679–684.

[39] R. Bellman, *Directions of Mathematical research in nonlinear circuit theory*, PGCT Trans., to appear.

[40] M. Krein and M. A. Rutman, *Linear Operators Leaving Invariant a Cone in Banach Space*, Amer. Math. Soc. Translation No. 26, 1950.

[41] T. E. Harris, *Branching Processes*, Ergebnisse der Math., to appear.

[42] R. Bellman, Introduction to Matrix Analysis, McGraw-Hill Book Co., Inc., New York, 1959.

[43] R. Bellman, I. Glicksberg, and O. Gross, *On the "bang-bang" control problem*, Q. Appl. Math., vol. 14 (1956), pp. 11–18.

[44] J. P. LaSalle, *On time optimal control systems*, Proc. Nat. Acad. Sci. USA, vol. 45 (1959), pp. 573–577.

[45] R. Bellman and J. M. Richardson, *On the application of dynamic programming to a class of implicit variational problems*, Q. Appl. Math., vol. 17, 1959, pp. 231–236.

[46] *Studies in Game Theory—I, II, III*, Annals of Math. Studies, Princeton University Press, Princeton, N. J.

[47] R. Bellman and R. Kalaba, *A Mathematical theory of adaptive control processes*, Proc. Nat. Acad. Sci. USA, vol. 45, 1959, pp. 1288–1290.