SOME ASPECTS OF ADAPTIVE CONTROL PROCESSES

By Robert Kalaba

1. Introduction

In various types of control processes arising in such diverse fields as engineering of automata, economics, biology, and communication, control devices are called upon to function under various conditions of uncertainty regarding underlying physical processes and environmental conditions. These conditions may range from rather detailed knowledge, on the one hand, to complete ignorance on the other. As the process unfolds, however, with control decisions being made all the while in an effort to obtain optimal system performance, additional information may become available to the control device concerning the unknown aspects of the process.

Under these circumstances the controller—animate or inanimate—has the possibility of "learning" to improve its performance based upon experience; i.e., it may *adapt* itself to circumstances as it finds them.

The functional equation technique of dynamic programming ([2]) can be used to attack wide classes of problems involving the determination of optimal control policies for adaptive controllers ([14], [8], [9], [11]). The closely allied problems of formulating adaptive control problems in precise mathematical terms and of presenting feasible computational algorithms for determining solutions via high speed digital computation are to be discussed. A unique blend of the theories of differential equations and difference equations, of probability, and of the calculus of variations results.

In recent years, to be sure, the mathematical theory of automatic control processes has been developed extensively ([26]). (An early contribution is Maxwell's paper, [21].) Many problems concerning stability and optimization have been formulated and successfully resolved ([25]). Furthermore, learning processes have been studied from the statistical side ([15], [18], [24]). Nevertheless, many challenges still remain ([29], [22], [25]); only a modest start has been made to date.

A concrete example in the form of a system governed by the inhomogeneous Van der Pol equation with a random forcing term is discussed in later sections. First it is assumed that the statistical properties of a random forcing term are only partially known to the controller, which seeks to maintain the system near its unstable equilibrium state during the control period through a suitable choice of the system parameter as a function of time. Through observation of the random forces the controller learns to improve the quality of its performance ([11]).

In the second example the controller is not given precise knowledge of the objective of the control process, but infers this during the course of the process. This is, to be sure, a situation of common occurrence in economic, engineering and other control processes. Even giving a precise mathematical formulation offers a distinct challenge.

Lastly, some outstanding problems in the general field of adaptive control are sketched.

2. Generalities Concerning Automatic Control Processes

One of the most important applications of differential equations lies in the description of the development of physical processes. One frequently assumes that the rate of change of the state vector, x, of a given system is solely a function, $f(x)$, of the state x and that initially the system is in a state c. This leads to the investigation of the system of equations.

$$\frac{dx}{dt} = \dot{x} = f(x), \qquad x(0) = c, \qquad 0 \leq t \leq T. \tag{1}$$

In many situations, though, the behavior of the system in its native form described by the equation (1) may be adjudged unsatisfactory, so that exertion of control forces may be required in an effort to secure more satisfactory performance. The equations for the process may then become

$$\dot{x} = f(x, y), \qquad x(0) = c \tag{2}$$

where $y = y(t)$ is a control vector, usually subject to various constraints. A typical control problem might then consist in the determination of a control vector $y = y(t)$, $0 \leq t \leq T$, which minimizes some functional F of x and y, e.g.,

$$F(x, y) = \int_0^T G(x, y, t)\, dt + H(x(T)). \tag{3}$$

The temptation now is to view this as a problem in the calculus of variations and terminate matters there.* For a number of reasons, though, we do not choose to do this. In the first place, the constraints on y can be a source of complication. In the second place, the resolution of the Euler equations subject to two-point boundary conditions is troublesome, especially if $f(x, y)$ is not a linear function. Thirdly, there are no classical counterparts to the stochastic control processes we wish to consider below. Lastly, the determination of y as a function of t is not what is desired for purposes of feedback control: we do desire to know the appropriate control force to be exerted as a function of the current state of the system and the time remaining before the termination of the process.

3. An Adaptive Control Process

For purposes of concreteness, let us consider a system whose behavior is described by the well-known Van der Pol equation

* See, for example, Chapters 11 and 12 of O. Bolza, Vorlesungen über Variationsrechnung, Teubner, Leipzig, 1909.

$$\ddot{u} + \epsilon(u^2 - 1)\dot{u} + u = 0, \qquad 0 \leq t \leq T, \tag{1}$$
$$u(0) = c_1, \qquad \dot{u}(0) = c_2.$$

As is known ([20]), the origin in the phase plane is an unstable equilibrium point, and there exists a periodic solution toward which all trajectories in the phase plane tend (assuming $c_1^2 + c_2^2 \neq 0$). Let us suppose that these oscillations are undesirable and that the system parameter ϵ is a variable which may be selected by a controller during the course of the control process, subject to the restrictions

$$a \leq \epsilon(t) \leq b, \qquad \int_0^T (\epsilon(t) - a_1)^2\, dt \leq c, \tag{2}$$

in an effort to maintain the system near the equilibrium state. Furthermore, let us agree to measure the cost incurred during the control process by the functional $J[\epsilon(t)]$, where

$$J[\epsilon(t)] = \int_0^T (|\, u(s)\,| + |\, v(s)\,|)\, ds + \exp\,(|\, u(T)\,| + |\, v(T)\,|), \tag{3}$$
$$\dot{u}(s) = v(s).$$

This monstrous criterion is chosen to stymie any attempt at direct analysis. The first term measures the cost of deviation from equilibrium during the interval $(0, T)$, and the second measures the cost of terminal deviation. We wish to determine control policies which minimize the cost, subject to the restrictions in the inequalities (2).

To handle this problem (and to prepare for the introduction of adaptive control processes and the use of digital computers), let us first reformulate it in terms of a discrete-valued time variable. Assume that the interval $(0, T)$ is divided into N equal subintervals of length h so that

$$Nh = T. \tag{4}$$

The equations (1) are replaced by the system of first-order difference equations

$$u_{k+1} = u_k + v_k h$$
$$v_{k+1} = v_k - [\epsilon_k(u_k^2 - 1)v_k + u_k]h, \qquad k = 0, 1, 2, \cdots, N - 1, \tag{5}$$

with

$$u_0 = c_1, \qquad v_0 = c_2. \tag{6}$$

It is, as usual, understood that

$$u(kh) = u_k, \qquad v(kh) = v_k, \qquad k = 0, 1, 2, \cdots, N. \tag{7}$$

The restraints on $\epsilon(t)$ become

$$a \leq \epsilon_k \leq b, \qquad k = 0, 1, 2, \cdots, N - 1,$$
$$\sum_{k=0}^{N-1} (\epsilon_k - a_1)^2 h \leq c, \tag{8}$$

and it is desired to minimize the function

$$(9) \qquad J(\epsilon_0, \epsilon_1, \cdots, \epsilon_{N-1}) = \sum_{k=0}^{N-1} (|u_k| + |v_k|)h + \exp(|u_N| + |v_N|)$$

through appropriate choice of the variables $\epsilon_0, \epsilon_1, \cdots, \epsilon_{N-1}$. The dependence of J on $\{\epsilon_k\}$ is through the equations in (5).

In more realistic situations we are forced to recognize that the behavior of the system may be materially influenced by a variety of external influences, so that equations (5) may be replaced by the equations

$$(10) \qquad \begin{aligned} u_{k+1} &= u_k + v_k h \\ v_{k+1} &= v_k - [\epsilon_k(u_k^2 - 1)v_k + u_k]h + g_k h, \qquad k = 0, 1, 2, \cdots, N - 1. \end{aligned}$$

In the deterministic case $\{g_k\}$ will be known to the controller as a function of k. In the more complex stochastic case the variables $\{g_k\}$ will be random variables with known distribution functions. Finally, in the adaptive case, the forces g_k will be random variables with distribution functions which are initially unknown, but which become more and more precisely determined as the process unfolds.

Let us pass directly to a discussion of the adaptive control process, [3]. An essential feature of the discussion is that the state of the system at any particular time involves not only a description of the physical state (displacement and velocity), but also a description of the state of the controller's knowledge concerning the external influences, the information pattern. In addition, it is clear that provision has to be made for characterizing the transformations on this state of knowledge in the course of the process as facts concerning the partially unknown external influences are accumulated and assimilated.

In its fullest aspects, we are here concerned with constructing a mathematical theory for one of the fundamental conundrums of experimental and theoretical science: construction of a physico-mathematical theory, discovery of additional physical facts, modification of the theory, etc. For a general discussion, see [8], [12]. In this generality it seems unlikely that we shall ever possess a thoroughly satisfactory theory of such processes; certainly in the foreseeable future they will provide a challenging series of problems to the mathematician. We shall show how a start can be made in their treatment.

We shall limit ourselves to a consideration of the problem in which the external influence g_k is a random variable for which

$$(11) \qquad \begin{aligned} \text{Prob}\{g_k = 1\} &= p \\ \text{Prob}\{g_k = -1\} &= 1 - p, \quad k = 0, 1, 2, \cdots, N - 1, \end{aligned}$$

where the precise value of the parameter p is unknown to the controller. We shall assume that these disturbances are independent random variables. It is to be expected, though, that in the course of the process the controller will be able to form increasingly reliable estimates of the parameter p, through observation and use of an estimator $p_{m,n}$ given by

$$p_{m,n} = \frac{(r + m)}{(r + m) + (s + n)}, \tag{12}$$

where

(13)
m = observed number of positive forces
n = observed number of negative forces
r, s = two nonnegative parameters which determine the initial estimate of p and its trustworthiness.

See [4], [12], and [11] for further discussion of this Bayes approach.

It is true, of course, that use of an estimator involves loss of information about the observed sequence $\{g_k\}$ (runs, etc.); there is, however, a great simplification in the description of what constitutes the controller's state of information, and in how this is transformed from time to time in the course of the process. In more general problems it is not so simple to describe the controller's state of knowledge and how it changes in the light of new information. The great task is to find reasonably simple descriptions of the information pattern, involving few parameters, which at the same time make it possible for the controller to make nearly optimal control decisions. Clearly, sufficient statistics will play a significant role, ([23], [16]).

Lastly, let us make the *assumption* that the controller is to regard all estimates as true values until further information is obtained. We can now proceed to the analytical formulation and computational solution.

4. Functional Equations

Let us motivate our procedure by reiterating that we do not wish to determine an optimal value of ϵ in terms of k and a given initial state; rather, we wish to determine a best value of ϵ in terms of the current state of the system (original system plus controller) and the time. This is the essence of the notion of feedback control, [9]. For deterministic processes this is a matter of convenience; for stochastic and adaptive processes it is essential.

With this end in mind we embed the original control process within a class of processes beginning at times $k = 0, 1, 2, \cdots, N - 1$, with the system in an arbitrary state and express the relationships among the members of this class. The state is a five-dimensional vector (c_1, c_2, c, m, n), where

(1)
c_1 = displacement
c_2 = velocity
c = constraint constant of equation (3.8)
m = observed number of positive random forces
n = observed number of negative random forces.

If the system is in the above state and the decision to have $\epsilon = z$ is made, then an immediate cost of $(|c_1| + |c_2|)h$ is incurred, and, depending on the sign of the random force g, one of two possible system states S_+ or S_-, results at time h later, where

$$S_+ = (c_1 + c_2h, c_2 - [z(c_1^2 - 1)c_2]h + h, c - (z - a_1)^2h, m + 1, n) \tag{2}$$

and

$$S_- = (c_1 + c_2h, c_2 - [z(c_1^2 - 1)c_2]h - h, c - (z - a_1)^2h, m, n + 1), \tag{3}$$

the first with estimated probability $p_{m,n}$, the second with estimated probability $1 - p_{m,n}$. With the system in the new state at the later time, a new choice of ϵ has to be made—a decision problem of the same type as the previous one.

Let us now introduce the functions

$$f_k(c_1, c_2, c, m, n) = \text{the estimated expected cost during the last } k \text{ stages of the control process in which the system is in state } (c_1, c_2, c, m, n) \text{ at time } (N - k)h, \text{ and using an optimal control policy, } k = 1, 2, \cdots, N - 1. \tag{4}$$

In taking expected values we assume that current estimates of statistical parameters are to be used as true values in the absence of additional information, which accounts for our use of the phrase "estimated expected cost." We shall determine these functions recursively, beginning with f_1, making use of the Bellman principle of optimality, ([2]). At the same time we shall determine the desired optimal control decisions. For f_1 we have

$$\begin{aligned} f_1(c_1, c_2, c, m, n) = {} & (| c_1 | + | c_2 |)h + \operatorname*{Min}_z \{p_{m,n} \exp [| c_1 + c_2h || \\ & + | c_2 - [z(c_1^2 - 1)c_2 + c_1]h + h |] \\ & + (1 - p_{m,n}) \exp \{| c_1 + c_2h | + | c_2 \\ & - [z(c_1^2 - 1)c_2 + c_1]h - h |\}\}, \end{aligned} \tag{5}$$

where the minimization is over z-values satisfying the conditions

$$a \le z \le b \qquad (z - a_1)^2h \le c. \tag{6}$$

For the process originating at time $(N - k)h$ we have

$$\begin{aligned} & f_k(c_1, c_2, c, m, n) \\ & = (| c_1 | + | c_2 |)h + \operatorname*{Min}_z \{p_{m,n}f_{k-1}(S_+) + (1 - p_{m,n})f_{k-1}(S_-)\}, \end{aligned} \tag{7}$$

for $k = 1, 2, \cdots, N$, where S_+ and S_- are as in equations (2) and (3) and z is restrained by the inequalities (6). Solution is to be effected by carrying out the indicated minimizations, first calculating the function f_1, then f_2, and so on.

The values of z as functions of k, c_1, c_2, c, m, and n, which do the minimizing, constitute the desired optimal control decisions for all states of the system at all times.

It is envisaged that solution will be achieved through use of high speed digital computing machines, the minimizations being achieved through use of straightforward search techniques. Observe that the original minimization problem has been reduced to a sequence of N simpler minimization problems. Since the solution in closed form of problems of the type under discussion here can be achieved only in rare cases (linear systems, quadratic criteria, as in [16], [19]) we are forced to turn to the study of computational techniques.

It is interesting to note that the embedding technique can be used in attacking a variety of other types of problems. See [13] and [10], where many other references are provided.

5. Computational Considerations

Unfortunately, it is not true that the sequence of functions $f_k(c_1, c_2, c, m, n)$, $k = 1, 2, 3, \cdots, N$, can be determined numerically in any routine fashion. The difficulty lies in the fact that in the determination of the function f_k, the function f_{k-1}, which depends on five variables, must be stored in the computing machine's memory. If each variable assumes just 10^2 values, then in all 10^{10} functional values have to be stored, a number well beyond the 10^4 or 10^5 words of high-speed storage available in current machines. This is the "curse of dimensionality" difficulty discussed by Bellman in [2]. Let us then turn to the important question of reduction of the computational difficulties in the determination of the sequence of functions $\{f_k\}$.

We observe that use of a Lagrange multiplier can effectively eliminate one variable, ([5]). We introduce the new cost function

$$\sum_{k=0}^{N-1} (|u_k| + |v_k|)h + \exp(|u_N| + |v_N|) + \lambda \sum_{k=0}^{N-1} (\epsilon_k - a_1)^2 h, \tag{1}$$

where λ is a positive parameter. For each choice of λ we now have to compute a sequence of functions of four variables $f_k(c_1, c_2, m, n)$), the dependence on c having been eliminated. The larger λ, the smaller will be the sum $\sum_{k=0}^{N-1} (\epsilon_k - a_1)^2 h$, which is an aid in determining the appropriate value of λ, the one which causes the last of inequalities (3.8) to be just satisfied. The recurrence relations become

$$\begin{aligned} f_k(c_1, c_2, m, n) = {} & (|c_1| + |c_2|)h + \min_{a \le z \le b} \{\lambda(z - a_1)^2 h \\ & + p_{m,n} f_{k-1}(c_1 + c_2 h, c_2 - [z(c_1^2 - 1)c_2 + c_1]h + h, m + 1, n) \\ & + (1 - p_{m,n}) f_{k-1}(c_1 + c_2 h, c_2 - [z(c_1^2 - 1)c_2 + c_1]h - h, m, n + 1\} \end{aligned} \tag{2}$$

for $k = 1, 2, \cdots, N$, and

$$\begin{aligned} f_1(c_1, c_2, m, n) = {} & (|c_1| + |c_2|)h + \min_{a \le z \le b} \{\lambda(z - a_1)^2 h \\ & + p_{m,n} \exp [|c_1 + c_2 h| + |c_2 - [z(c_1^2 - 1)c_2 + c_1]h + h|] \\ & + (1 - p_{m,n})[\exp [|c_1 + c_2 h| + |c_2 - [z(c_1^2 - 1)c_2 + c_1]h - h|]]\}. \end{aligned} \tag{3}$$

It would now be desirable to determine the sequences $\{f_k\}$ for several values of the parameter λ, calculate the corresponding values of $\sum_{k=0}^{N-1} (\epsilon_k - a_1)^2$, and in this way approximate the solution of the problem for which

$$\sum_{k=0}^{N-1} (\epsilon_k - a_1)^2 h = c, \tag{4}$$

performing at the same time a valuable parameter study. Unfortunately, though, for currently available computers the problem is still of unmanageable size.

Further reductions are possible. Since the difficulty is essentially one of storage of a function of four variables defined on a certain bounded domain, one key to its solution is use of a slightly more sophisticated method of storage, one not involving storage of the functional values themselves. This can be achieved by approximating the functions by polynomials, for example, so that essentially only the coefficients of the polynomials need be stored. Of course, this implies that values of the functions will have to be computed as they are needed and not merely recalled from memory, which will add greatly to the computing time. The net result is that the problem may be brought from the realm of the incomputable to the realm of the computable. Memory requirements are reduced at the expense of increased times of computation. Much has been done along these lines [1], [3], [16]), but much remains to be done.

But still other radically different possibilities present themselves for consideration, primarily based on adaptation of Picard's classical method of successive approximation ([3]). To illustrate: it is known that if the performance of a system is described by linear equations and the criterion functional is a function of the final state involving only k of the state variables, then the process can be analyzed as a k-dimensional process, rather than one of the higher dimensionality ([6]). Through appropriate quasi-linearization of a given set of nonlinear equations, similar to the method used in [17], we may envisage solution through successive approximations, each approximation computation being a major computation in itself. See also [7].

Another approach is suggested by the realization that asking for optimal policies may be asking for too much. A more modest requirement is that we devise methods for merely improving on a given control policy, if it can be done. Of course, even when this is accomplished, the usual difficulties involving relative versus absolute optimal arise. Further discussion is presented in Bellman's monograph ([13]).

We note lastly that when $m + n$ becomes large, we may have confidence that the estimate $p_{m,n}$ is close to the true probability p. This being the case, two variables, c_1 and c_2, serve to specify the state of the system. The functional equations then involve functions of two variables,

$$\begin{aligned} f_k(c_1, c_2) = (| c_1 | + | c_2 |)h + \operatorname*{Min}_{a \le z \le b} \{&\lambda(z - a_1)^2 h + p_{m,n} f_{k-1}(c_1 + c_2 h, c_2 \\ &- [z(c_1^2 - 1)c_2 + c_1]h + h) + (1 - p_{m,n}) f_{k-1}(c_1 + c_2 h, c_2 \\ &- [z(c_1^2 - 1)c_2 + c_1]h - h)\}, \end{aligned} \tag{5}$$

for $k = 2, 3, \cdots, N$, and

$$f_1(c_1, c_2) = (|c_1| + |c_2|)h + \underset{a \leq z \leq b}{\text{Min}} \{\lambda(z - a_1)^2 + p_{m,n} \exp[| c_1 + c_2 h |$$

$$(6) \qquad + | c_2 - [z(c_1^2 - 1)c_2 + c_1]h + h |] + (1 - p_{m,n}) \exp [| c_1$$

$$+ c_2 h | + | c_2 - [z(c_1^2 - 1)c_2 + c_1]h - h |]\}.$$

6. Unknown Criteria

Let us now sketch a treatment of a process where uncertainty of a different type is involved, where the very purpose of the process is unknown to the controller. In the course of the process, though, the controller obtains more and more information about the precise nature of the objective function. The process to be discussed is one in which the controller seeks to have the system in the "right place at the right time," but does not know precisely where the "right place" is. Various problems involving vehicle guidance are of this genre.

To render these ideas more precise, let us consider a system, the physical state of which is governed by the equations

$$(1) \qquad x_{n+1} = x_n + v_n h, \qquad x_0 = c_1$$

$$v_{n+1} = v_n - [\epsilon_n(1 - x_n^2(v_n + x_n]h, \qquad v_0 = c_2,$$

which hold for $n = 0, 1, 2, \cdots, N - 1$. We shall suppose, in the deterministic version, that it is desired to have the process terminate at time $t = Nh$ with $x_N = c_3$.

More precisely, since this will in general not be possible, let the problem be that of selecting a control sequence $(\epsilon_0, \epsilon_1, \cdots, \epsilon_{N-1})$ in such a way as to minimize the expression

$$(2) \qquad J = | x_N - c_3 |.$$

If, next, the controller does not know the value of c_3 in advance, but only knows that

$$(3) \qquad \text{Prob}\,\{c_3 = r_1\} = p$$

$$\text{Prob}\,\{c_3 = r_2\} = 1 - p,$$

where the precise value of p is known, then control may be exerted in such a way as to minimize the expected value of J, $E\{J\}$, where

$$(4) \qquad E\{J\} = p \,| x_N - r_1 | + (1 - p)| x_N - r_2 |,$$

for

$$(5) \qquad a \leq \epsilon_n \leq b, \qquad n = 0, 1, 2, \cdots, N - 1.$$

Lastly, in the adaptive case, we assume that the precise value of p is unknown initially. During the course of the process, though, we assume that the controller is able to modify its a priori estimate of p through sampling the population from which the final selection of c_3 is to be made. As before, if the initial estimate of p is $e_1/(e_1 + e_2)$, after observing m r_1's and n r_2's the estimate of p is taken to be $p_{m,n}$, where

$$p_{m,n} = \frac{e_1 + m}{(e_1 + m) + (e_2 + n)}. \tag{6}$$

The objective is to minimize the estimated expected value of J. For this case the functional equations become

$$\begin{aligned} f_{k+1}(c_1, c_2, m, n) = \underset{a \le z \le b}{\text{Min}} \{ & p_{m,n} f_k(c_1 + c_2 h, c_2 \\ & - [z(c_1^2 - 1)c_2 + c_1]h, m + 1, n) + (1 - p_{m,n}) f_k(c_1 + c_2 h, c_2 \\ & - [z(c_1^2 - 1)c_2 + c_1]h, m, n + 1)\} \end{aligned} \tag{7}$$

for $k = 1, 2, \cdots, N - 1$, and

$$f_1(c_1, c_2, m, n) = \underset{a \le z \le b}{\text{Min}} \{p_{m,n} | c_1 + c_2 h - r_1 | + (1 - p_{m,n}) | c_1 + c_2 h - r_2 |\}, \tag{8}$$

where one observes that the term in brackets in the right hand side of equation (8) is independent of z. Details will be given in a later paper.

7. Discussion

The objective of a theory of adoptive control processes is the determination of optimal control decisions under conditions of incomplete information where, however, learning takes place during the unfolding of the process. In the first process discussed, the controller's lack of knowledge was limited to uncertainty regarding the state into which the system is transformed as the result of any particular control decision. Ignorance, though, can manifest itself in many other ways. The controller may not know the precise physical state of the system, the allowable set of decisions, or the duration of the process. As we have seen, not even the objective of the process may be precisely known. All of these possibilities require intensive investigation. And interesting questions arise at all levels.

A great many problems arise in the precise mathematical formulation of adaptive control problems, though the functional equation imbedding technique provides certain guide lines along which to proceed.

At the analytical level we face a host of queries. These range from questions of convergence of solutions of the discrete-time problems to solutions of the continuous-time problems, to the determination of various structural properties of solutions of nonlinear functional equations.

At the computational level we face the problems of high dimensionality, stability, and so on. It is interesting to note that the limited capabilities of modern computers force us to be quite ingenious in our formulations and analysis before the *problem* is submitted for computational solution. Thus the use of computers—in the popular mind so suggestive of that which is routine, trite, and unimaginative—necessitates highly imaginative treatment by the analyst at all levels. Furthermore each step forward raises additional questions giving added meaning to Poincaré's statement, "Solution of a mathematical problem is a phrase of indefinite meaning."

RAND Corporation, Santa Monica, California

References

[1] M. Aoki, Ph.D. thesis, University of California, to appear.

[2] R. Bellman, Dynamic Programming, Princeton Univ. Press, Princeton, N. J., 1957.

[3] R. Bellman. Adaptive Control; A Guided Tour, Princeton Univ. Press, Princeton, N. J., 1960.

[4] R. Bellman, *A Problem in the sequential design of experiments*, Sankhyā, v. 16 (1956) pp. 221–229.

[5] R. Bellman, *Dynamic programming and Lagrange multipliers*, Proc. Nat. Acad. Sci. USA, v. 42 (1956), pp. 767–769.

[6] R. Bellman, *Some new techniques in the dynamic programming solution of variational Problems*, Quart. of App. Math., v. 16 (1958), pp. 295–305.

[7] R. Bellman, *Terminal control, time lags, and dynamic programming*, Proc. Nat. Acad. Sci. USA, v. 43 (1957), pp. 927–930.

[8] R. Bellman and R. Kalaba, *Dynamic programming and adaptive processes: Mathematical foundation*, IRE Trans. on Automatic Control, v. 5 (1960), pp. 5–10.

[9] R. Bellman and R. Kalaba, *Dynamic programming and feedback control*, Proceedings of the First International Congress on Automatic Control, Moscow, June, 1960.

[10] R. Bellman and R. Kalaba, *Functional equations in adaptive processes and random transmission*, IRE Trans. on Circuit Theory, v. CT-6 (Special Supplement, May, 1959), pp. 271–282.

[11] R. Bellman and R. Kalaba, *On adaptive control processes*, 1959 IRE National Convention Record, Part 4, pp. 3–11, August 1959.

[12] R. Bellman and R. Kalaba, *On communication processes involving learning and random duration*, 1958 IRE National Convention Record, Part 4, pp. 16–21, July 1958.

[13] R. Bellman and R. Kalaba, *On the principle of invariant imbedding and propagation through inhomogeneous media*, Proc. Nat. Acad. Sci. USA, v. 42 (1956), pp. 629–632.

[14] R. Bellman and R. Kalaba, *A mathematical theory of adaptive control processes*, Proc. Nat. Acad. Sci. USA, v. 45 (1959), pp. 1288–1290.

[15] M. M. Flood, *On game-learning theory and some decision-making experiments*, Decision Processes, edited by R. M. Thrall, C. H. Coombs, and R. L. Davis, John Wiley and Sons, Inc., New York, 1954.

[16] M. Freimer, Ph.D. Thesis, Harvard University, to appear.

[17] R. Kalaba, *On nonlinear differential equations, the maximum operation, and Monotone Convergence*, Jour. Math. and Mech., v. 8 (1959), pp. 519–574.

[18] S. Karlin, R. Bradt and S. Johnson, *On sequential designs for maximizing the sum of n observations*, Ann. Math. Stat., v. 27 (1956) pp. 1061–1074.

[19] J. D. R. Kramer, unpublished.

[20] S. Lefschetz, Differential Equations: Geometric Theory, Interscience Publishers, Inc., New York, N. Y., 1957.

[21] J. C. Maxwell, *On governors*, Proc. Royal Soc., v. 16 (1868), pp. 270–283.

[22] B. McMillan, *Where do we stand?*, IRE Trans. on Information Theory, v. IT-3 (1957) pp. 173–174.

[23] A. M. Mood, Introduction to the Theory of Statistics, McGraw-Hill Book Company, Inc., New York, 1950.

[24] H. Robbins, *Some aspects of the sequential design of experiments*, Bull. Amer. Math. Soc., v. 58 (1952) pp. 527–535.

[25] R. P. Stromer, *Adaptive or self-optimizing control systems—A bibliography*, IRE Trans. on Automatic Control, v. AC-4 (1959), pp. 65–68.
[26] J. Truxal, Automatic Feedback Control System Synthesis, McGraw-Hill Book Company, Inc., New York, N. Y., 1955.
[27] A. Wald, Sequential Analysis, John Wiley and Sons, Inc., New York, 1947.
[28] N. Wiener, Cybernetics, John Wiley and Sons, Inc., New York, 1948.
[29] L. Zadeh, *What is optimal?* IRE Transactions on Information Theory, v. IT-4 (1958), p. 3.