THE CRITICAL CASE IN DIFFERENTIAL EQUATIONS

By Solomon Lefschetz

Introduction

Real analytical systems of ordinary differential equations with an isolated critical point (as usual placed at the origin) are of particular interest both in theory and in countless applications. The case where the coefficient matrix K of the first degree terms has no characteristic roots with zero real parts—the general case—has been amply treated in the literature, particularly from the standpoint of stability. Far less is known when some of the characteristic roots deviate from the above norm.

Liapunov ([4]) has fully dealt with the following two cases: (a) a single characteristic root is zero; (b) there is a pair of pure complex roots. In both situations he assumed that the other characteristic roots have negative real parts.

The same problem and its generalization have been treated extensively by several Soviet authors, above all by the late Malkin. Such cases are called by them "critical." There are several papers by Malkin and considerable material in his book ([5]). More material is also to be found in two very recent Soviet books by Krassovskii ([2]) and by Zubov ([6]). See also Lefschetz' [3].

While the Soviet writings on this topic are abundant they cannot be said to shine with excessive clarity. It has, at least, been our experience that it is very difficult to say what has or has not been accomplished. We have, therefore, undertaken the modest task of treating a special but important case with some care, by comparatively simple procedures, to see how far one can go in that manner.

We consider then the case where the matrix K has a certain number of characteristic roots zero, with simple invariant factors, the other roots having negative real parts. That is, the matrix K can be transformed to the type $\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$ where the zeros represent matrices and the characteristic roots of the matrix A have negative real parts. This excludes for example the case where the matrix at the upper left corner has zero roots but not all terms zero. For instance, if it is of order two it cannot be $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. While in the cases examined by Liapunov and Malkin a complete, or nearly complete, description of the stability situation was attainable, we have been able only to give a theorem on asymptotic stability and one on instability.

Two appendices terminate the paper. The first is merely a rapid résumé of the required theorems of Liapunov and Četaev. The second gives a fairly complete proof of a special case of a basic result due to Zubov and utilized in the text.

Notations and designations. The components of a vector u will be designated by Greek subscripts: u_λ, u_μ, $\cdots$.

If u, v are two vectors we write $[u]_k$ = a vector whose components are real power series convergent near $u = 0$ and beginning with terms of degree $\geqq k$;

$(v)\ [u]_k$ = the same but with coefficients power series in v convergent near $v = 0$;

$[v]_j[u]_k$ is like the preceding save that the v series are $[v]_j$.

All vectors are usually column-vectors, that is, one column matrices. If x is such a vector, x' stands for the row-vector with the same components (transpose of the matrix x). Thus $x'y$ is the inner product of x by y.

If $f(x)$ is a scalar differentiable function of the vector x then grad $f = \partial f/\partial x$ is the row-vector $(\partial f/\partial x_\lambda)$.

The following abridged type of locution will be utilized. Let $X(x)$ be a vector-function of the vector x. We will say for example that X is a power series in x, or holomorphic in x at $x = 0$ whenever the components X_λ of X are power series in those of x behaving as stated. Also say $X = [x]_k$ will mean that the X_λ are $[x]_k$, etc.

In a moment we will have occasion to consider in an n-space vectors of different dimensions: p and q, $p + q = n$. We shall adopt the following notations:

y, Y, g, G, γ will denote p-vectors or vector-functions;

z, Z will denote q-vectors or vector-functions.

A *stable* matrix is one whose characteristic roots all have negative real parts.

Norms, such as $\| x \|$, are always assumed Euclidean.

A property of real analytic equations. Consider the real vector equation

$$(*) \qquad Z(y, z) = 0$$

where $Z(y, z) = [y, z]_1$ and the Jacobian determinant

$$(**) \qquad \left| \frac{\partial Z}{\partial z} \right| = \left| \frac{\partial Z_\lambda}{\partial z_\mu} \right| \neq 0$$

for $y = 0$, $z = 0$. The dimensions as indicated by the notations are q for the vectors Z, z and p for the vector y.

In view of $(**)$ the system $(*)$ has a unique solution $z(y)$ holomorphic and zero at $y = 0$. We wish to prove the almost obvious

LEMMA. *The solution $z(y)$ is real.*

The proof is quite simple. The component series $z_\lambda(y)$ are obtained by assuming indeterminate coefficients, substituting in $(*)$ and making all resulting coefficients of $y_1, \cdots, y_p$ equal to zero. This yields a succession of real linear equations in the terms of each successive degree, and they are known to have a unique solution each time. These successive solutions are therefore real. It follows that the solution $z(y)$ is itself real; that is, the components $z_\lambda(y)$ are real power series in $y_1, \cdots, y_p$.

This lemma will enable us to disregard reality conditions throughout the paper.

I. First reduction (Liapunov)

1. Consider the n-vector analytical system

$$(1.1)\qquad \begin{aligned} \dot{x} &= Kx + X(x) \\ X(0) &= 0, \quad X(x) = [x]_2\,, \end{aligned}$$

where x, X are n-vectors and K is a constant matrix with p characteristic roots zero, and the rest with negative real parts. Moreover, the zero roots have only linear invariant factors. As a consequence ([3], p. 329) one may apply a real linear transformation of coordinates reducing the basic system (1.1) to the form

$$(1.2)\qquad \begin{aligned} &\text{(a)}\quad \dot{y} = Y(y, z) \\ &\text{(b)}\quad \dot{z} = Az + Z(y, z), \end{aligned}$$

where the vectors have dimensions already stated and $Y(y, z)$, $Z(y, z)$ are $[y, z]_2$. Note that the origin is a critical point. *We assume explicitly that it is an isolated critical point.*

Regarding A, it is a constant, stable, $q \times q$ matrix whose characteristic roots are the non-zero characteristic roots of K. Thus K is equivalent to the $n \times n$ matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

Note that as a consequence of this final form of K we also have $Z(y, 0) = [y]_k$, $k > 1$. That is $Z(y, z)$ contains no terms of the first degree in y alone.

2. We shall apply to the system (1.2) a finite set of transformations regular at the origin. They will be so selected as to prepare the system (1.2) for the application of the Liapunov theorems. All that one needs to remember is that *such transformations do not alter the stability properties of the origin*, an observation which will not be repeated.

The first transformations is inevitable and goes back to Liapunov. To define it one must first take the system

$$(2.1)\qquad A\zeta + Z(y, \zeta) = 0$$

in the unknown q-vector ζ. This is an analytic system of q equations in the components ζ_λ of ζ whose left hand sides vanish for $\zeta = 0, y = 0$. For $\zeta = 0, y = 0$ the Jacobian as to the ζ_λ is the determinant $|A|$ which is $\neq 0$ since A is stable. Hence there is a unique solution $\zeta(y)$ holomorphic at the origin and it is real (lemma of the Introduction).

One may make an important observation. Write

$$\zeta(y) = \zeta_s(y) + \zeta_{s+1}(y) + \cdots,$$

where $\zeta_h(y)$ is a vector whose components are forms of degree h in $y_1, \cdots,$

y_p. By substituting in (2.1) we obtain

$$A\{\zeta_s(y) + \zeta_{s+1}(y) + \cdots\} = -Z(y, \zeta_s(y) + \zeta_{s+1}(y) + \cdots)$$

or

$$\zeta_s(y) + \zeta_{s+1}(y) + \cdots = -A^{-1}Z(y, \zeta_s(y) + \cdots). \tag{2.2}$$

Since $Z(y, z) = [y, z]_2$, $Z(y, 0) = [y]_k$, $k > 1$, we see that at the right in (2.2) the terms of degree $<k$ vanish. Hence $\zeta(y) = [y]_k$.

One defines now the Liapunov transformation as

$$y \to y, \qquad z \to z^* + \zeta(y). \tag{2.3}$$

The resulting new system is

$$\text{(2.4)}\quad \begin{aligned} &\text{(a)}\quad \dot{y} = Y(y, z^* + \zeta(y)) \\ &\text{(b)}\quad \dot{z}^* = Az^* + Z(y, z^* + \zeta(y)) - Z(y, \zeta(y)) - \frac{\partial \zeta}{\partial y} Y(y, z^* + \zeta(y)). \end{aligned}$$

Recall here that $\partial\zeta/\partial y$ is the Jacobian matrix $(\partial\zeta_\lambda(y)/\partial y_\mu)$.

3. Two possibilities present themselves at the present stage.

(a) $Y(y, \zeta(y)) \equiv 0$. Then every point of the whole p-space $z = 0$ is a critical point. Since $p > 0$, this means that the origin is not an isolated critical point. Under our hypotheses this case is ruled out.

(b) $Y(y, \zeta(y)) \not\equiv 0$. For convenience write henceforth z for z^*. The new system may then be written

$$\text{(3.1)}\quad \begin{aligned} &\text{(a)}\quad \dot{y} = G(y, z) + \gamma_0(z) + \gamma_1(y, z) + \cdots + \gamma_{N-1}(y, z) \\ &\text{(b)}\quad \dot{z} = Az + Z(y, z), \end{aligned}$$

with the degree properties $G(y, 0)$ and $G(y, z)$ are both $[y]_N$, where $N > 1$; $\gamma_0(z) = [z]_2$, $\gamma_k(y, z) = [y]_k(z)_1$, $Z(y, z) = [y, z]_2 = [y]_1[z]_1$, $Z(y, 0) = [y]_{N+1}$. The last affirmation rests upon the earlier remark that $\zeta(y) = [y]_k$, $k > 1$.

Let $g(y)$ be the collection of terms of degree N (lowest degree) in $G(y, 0)$. While further simplifying transformations will be applied they will not affect $g(y)$. This enables us to state already our asymptotic stability result.

THEOREM (3.2). *If $y = 0$ is asymptotically stable for the auxiliary system*

$$\dot{y} = g(y), \tag{3.2a}$$

(it can only happen if N is odd), then the origin $y = 0$, $z = 0$ is likewise asymptotically stable for the complete system (3.1) *(and hence also for the initial system* (1.1)*).*

II. Second reduction

4. We shall now apply a (finite) series of regular transformations so designed as to have every $\gamma_k(y, z) = [y]_k[z]_2$, that is so as to increase by unity the lowest degree in z in $\gamma_k(y, z)$, all this of course for $1 \leqq k \leqq N - 1$.

Suppose that our purpose has already been accomplished for $\gamma_1, \cdots, \gamma_{k-1}$, $k < N$. To extend the reduction to γ_k apply the real regular transformation

$$y = \bar{y} + \gamma_k^*(\bar{y}, z) \tag{4.1}$$

where γ_k^* has for components forms of degree k in those of $\bar{y}$, with coefficients linear in z. We need some knowledge of the inverse of the transformation which is also real. Let it be

$$\bar{y} = y + \gamma_s^{**}(y, z) + \gamma_{s+1}^{**}(y, z) + \cdots$$

where γ_h^{**} has for components forms of degree h in y. By substitution in (4.1) and identification we find at once that $s = k$ and that $\gamma_k^{**} = -\gamma_k^*(y, z)$, while $\gamma_{k+s}^{**} = [z]_1$ for all s. Thus the inverse has the form

$$\bar{y} = y - \gamma_k^*(y, z) + \gamma_{k+1}^{**}(y, z) + \cdots. \tag{4.2}$$

The information that this expression conveys is sufficient for our purpose.

The two expressions (4.1) and (4.2) make it plain that if an expression is $[y]_k[z]_h$, upon applying (4.1) it becomes $[\bar{y}]_k[z]_h$ and conversely. On the strength of this remark one verifies readily that the transformation (4.1) does not affect the form of (3.1b) and in particular it does not affect the matrix A. We may, therefore, assume that (3.1b) is already expressed in the new variables. As for (3.1a) the only terms changed in form are the γ_j, $j \geqq k$. In particular, let $\gamma_{k\lambda}(z)$, $\gamma_\lambda^*(z)$ denote the λ-th components of γ_k, γ^*. Let $b' \cdot z$ denote the known coefficient of

$$y_1^{r_1} \cdots y_p^{r_p}, \qquad r_1 + \cdots + r_p = k$$

in $\gamma_{k\lambda}$ and $c' \cdot z$ the same (unknown) in γ_λ^*. In the new γ_k the same will be the difference $b' \cdot z - c' \cdot Az$. To annul it we merely need to take $b' = c'A$, or $c = A'^{-1}b$, which can always be done since $| A | \neq 0$. If we operate likewise for every combination

$$y_1^{r_1} \cdots y_p^{r_p}, \qquad \sum r_i = r,$$

we will obtain γ^*, and our purpose will be accomplished for γ_k.

Proceeding in this manner we can arrive at the form (3.1a) with $\gamma_k(y, z) = [y]_k[z]_2$.

III. Proof of the asymptotic stability theorem

5. All that is necessary is to prove the theorem for the system (3.1) reduced in the manner just described.

Suppose then that (3.2a) is asymptotically stable. By Zubov's theorem (see Appendix II) there exist two homogeneous functions $V_0(y)$, $W_0(y)$ where V_0 is a positive definite quadratic form (see 15), W_0 is a negative definite form of degree $N + 1$ (hence N must be odd), and

$$\dot{V}_0 = W_0 \tag{5.1}$$

along the paths of (3.2a). That is

(5.2) $$W_0 = (\text{grad } V_0)\cdot g(y).$$

On the other hand, since A is a stable matrix, the system

(5.3) $$\dot{z} = Az$$

is asymptotically stable at $z = 0$. Hence (see Appendix I), there exists a positive definite quadratic form $V_1(z)$ such that along the paths of (5.3)

$$\dot{V}_1(z) = W_1(z)$$

is a negative definite quadratic form. Take now

$$V(y, z) = V_0(y) + V_1(z).$$

We will show that V behaves in accordance with Liapunov's theorem on asymptotic stability.

In the first place V is manifestly a positive definite function. We must show that the same holds for $-\dot{V}$.

6. A preliminary remark is required. Let $f(x)$ be a real homogeneous scalar continuous function of degree s in the n-vector x. Then $\varphi(x) = \dfrac{f(x)}{\| x \|^s}$ is a similar function of degree zero. Hence its values are those which f takes on $S(1)$, the sphere $\| x \| = 1$. Since $S(1)$ is compact, and φ is continuous on $S(1)$ it assumes its extreme values α, β. Hence

(6.1) $$\alpha\cdot \| x \|^s \leqq f(x) \leqq \beta\cdot \| x \|^s.$$

Note that if $f(x)$ is positive definite both α and β are positive.

7. Returning to our problem we must calculate $\dot{V}(y, z)$ along the paths of (3.1) near the origin. We find

$$\dot{V}(y, z) = \text{grad } V_0(y)\cdot(G(y, z) + Y(y, z)) + \text{grad } V_1(z)\cdot(Az + Z(y, z))$$

where all the quantities other than the V's are as in (3.1). We have then

(7.1) $$\begin{aligned}\dot{V}(y, z) = &\{\text{grad } V_0(y)\cdot g(y) + \text{grad } V_1(z)\cdot Az\}\\ &+ \{\text{grad } V_0(y)\cdot(G(y, z) - g(y))\\ &+ \text{grad } V_0(y)\cdot Y(y, z) + \text{grad } V_1(z)\cdot Z(y, z)\}.\end{aligned}$$

Let us show that terms in the second bracket are dominated by appropriate terms in the first.

(a) Since $G(y, z) - g(y)$ consists of terms of degree $N + 1$ in y alone plus terms of the form $[y]_N[z]_1$, it is dominated by $g(y)$. Hence grad $V_0(y)\cdot(G - g)$ is dominated by grad $V_0(y)\cdot g$.

(b) The terms of $Y(y; z)$ in z alone are $[z]_2$ hence those of grad $V_0(y)Y(y; z)$ are $[y]_1[z]_2$, and hence dominated by those of the definite quadratic form

$$\text{grad } V_1(z)\cdot Az.$$

(c) The terms of $Z(y; z)$ are either $[y]_{N+1}$ in y alone, $[z]_2$ in z alone or $[y]_1[z]_1$. Hence the terms of grad $V_1(z) \cdot Z(y; z)$ are dominated by those of grad $V_0(y) \cdot g(y)$ or of grad $V_1(z) \cdot Az$.

We conclude that the sign of $\dot{V}$ is that of the first bracket and is negative for $y \neq 0$ or $z \neq 0$ or both. To sum up then V and $-\dot{V}$ are positive definite in a suitably small neighborhood of the origin. Hence theorem (3.2) is a consequence of the asymptotic stability theorem of Liapunov (Appendix I).

IV. An instability theorem

8. The theorem that has just been proved operates on an odd degree N. An extension of the Liapunov proof of instability for the special case $p = 1$ will yield the result embodied in the theorem to follow.

Let then N be even. The components $g_1(y), \cdots, g_p(y)$ of $g(y)$ are forms of the even degree N.

INSTABILITY THEOREM (8.1). *If there exists a constant row-p-vector $\gamma \neq 0$ such that the scalar product*

$$g^*(y) = \gamma \cdot g = \sum \gamma_i g_i$$

is positive definite then the system (3.1), *and hence also* (1.1), *is unstable.*

Note first that if $g^*(y)$ exists then we may choose the vector γ with arbitrarily small modulus $\| \gamma \|$. We will utilize this property in a moment.

Since the reduction to the form (3.1) has nothing to do with the parity of N, we may still assume that we deal with that system. Define now

$$V(y, z) = \gamma \cdot y - V_1(z),$$

where $V_1(z)$ is the same as previously.

Then

$$\dot{V}(y, z) = \{\gamma \cdot G(y) - W_1(z) \cdot \gamma \cdot Q(z)\} + \cdots$$

where $Q(z)$ is the p-vector made up of the lowest degree terms in the $\gamma_0(z)$, and the dots represent terms which are small relative to the terms taken. Thus the bracket contains all the terms dominant as to sign.

Consider first the quadratic form $-W_1(z) + \gamma \cdot Q(z)$. The first term is positive definite. We may write

$$-W_1(z) + \gamma \cdot Q(z) = \| z \|^2 \left\{ \frac{-W_1(z)}{\| z \|^2} + \frac{\gamma^* Q(z)}{\| z \|^2} \right\}.$$

Now the bracket is the same as

$$(8.2) \qquad -W_1(z) + \gamma \cdot Q(z)$$

taken on the sphere of radius one. Let α be the least value of $-W_1(z)$ on the sphere. As observed earlier one may replace the vector γ by any proportional vector say $m\gamma$, $m > 0$. Thus (8.2) may be written

$$-W_1(z) + m\gamma \cdot Q(z).$$

Let β be the greatest value of $|\gamma \cdot Q(z)|$ on the sphere. Note that $\beta > 0$. Thus (8.2) will be at least as great everywhere on the sphere as $\alpha - m\beta$. Hence if we take say $m < \frac{\alpha}{2\beta}$, the expression (8.2) will be positive everywhere on the sphere and we will have (replacing $m\gamma$ by γ):

$$\dot{V}(y, z) = \{\gamma \cdot g(y) - W_1(z)\} + \cdots,$$

where the sign of the bracket is the sign of $V(y, z)$ for y, z small. In other words $\dot{V}$ will be positive definite in a certain closed neighborhood Ω of the origin in the y, z space.

On the other hand, V can manifestly assume the sign + in a certain subregion Ω_1 of Ω. All that is necessary for example is to observe this. Since $\gamma \neq 0$ one of its components say $\gamma_h \neq 0$. At the point $z = 0$, $y_1 = \cdots = y_{h-1} = y_{h+1} = \cdots = y_p = 0$, $y_h = e_h$, where $e_h \gamma_h > 0$, we will have $V > 0$. Hence the region $V > 0$ exists (in fact in the whole space). In that region in $\Omega \cdot \dot{V} > 0$ also, and so by Četaev's theorem (see Appendix I) our system is unstable and our theorem is proved.

V. The special cases $p = 1, 2, 3$

9. While these cases are not exceptional they permit further information regarding stability.

The case $p = 1$. This time y and the elements associated with it are all scalars. More explicitly the system (3.1) becomes

$$(9.1)\quad \begin{aligned} &\text{(a)}\quad \dot{y} = G(y, z) + Y(y, z) + \cdots \\ &\text{(b)}\quad \dot{z} = Az + Z(y, z) \end{aligned}$$

where y, G, Y, are all scalars and

$$G(y, z) = y^N(g_0(z) + yg_1(z) + \cdots), \qquad g_0(0) = g \neq 0.$$

This time we may state with Liapunov:

THEOREM (9.2). *The system with $p = 1$ is unstable unless N is odd and g is negative, and then it is asymptotically stable.*

The cases N even or odd must be considered separately.

N even. Let $V_1(z)$, $W_1(z)$ be the same quadratic forms as before and let $Q(z)$ be the collection of the terms of lowest degree in z alone in $Y(y, z)$. Take

$$V(y, z) = \epsilon(\alpha y - gV_1(z)),$$

where $\epsilon = \pm 1$, $\epsilon g > 0$ and $\alpha > 0$. Then

$$\dot{V} = \epsilon g\{\alpha y^N + \alpha Q(z) - W_1(z)\} + \cdots,$$

where the only significant terms as regards the sign are those in the bracket. As in No. 8 we may choose α so small that $\alpha Q(z) - W_1(z)$ is a positive definite quadratic form. Thus $\dot{V}$ will be definite positive in a certain region Ω around the origin. On the other hand in a certain subregion Ω_1 around the origin $V(y, z)$ can evidently be positive. Hence by Četaev's theorem we have instability.

N odd. This time we choose

$$V(y, z) = \frac{y^2}{2} - gV_1$$

so that

$$\dot{V}(y, z) = g\cdot\{y^{N+1} - W_1\} + \cdots .$$

If $g > 0$, $\dot{V}$ is definite positive in a certain Ω while $V > 0$ in a suitable Ω_1 so that we have instability. On the other hand if $g < 0$ both V and $-\dot{V}$ are positive definite in a suitable Ω and so we have asymptotic stability. This completes the proof of the theorem.

10. *The case* $p = 2$. This case has been discussed at some length by Malkin ([5], p. 407). We take it up from a somewhat different (and more geometric) point of view.

Consider then the system

$$\dot{x} = X_m(x, y), \qquad \dot{y} = Y_m(x, y) \tag{10.1}$$

where X_m, Y_m are real forms of degree m of the scalar variables x, y. We propose to determine necessary and sufficient conditions for the asymptotic stability of (10.1). This will provide, at all events, sufficient conditions for the asymptotic stability of the basic system (1.2) when $p = 2$.

Set

$$U_{m+1}(x, y) = xX_m + yY_m, \qquad V_{m+1}(x, y) = xY_m - yX_m, \qquad r^2 = x^2 + y^2, \tag{10.2}$$

so that U, V are forms of degree $m + 1$.

Then

$$r^2X = xU - yV, \qquad r^2Y = yU + xV. \tag{10.3}$$

From these relations we infer at once the following two properties:

(10.4) *U and V cannot both be identically zero.*

(10.5) *U and V cannot have a common real linear factor z.* For z would be a common linear factor of X and Y. Hence all the points of the line $z = 0$ would be critical and so the origin could not be asymptotically stable.

Passing now to polar coordinates r, θ and with an appropriate new time unit, still called t, the system (10.1) is replaced by

$$\begin{aligned} \dot{r} &= rU_{m+1}(\sin\theta, \cos\theta) = rU(\theta) \\ \dot{\theta} &= V_{m+1}(\sin\theta, \cos\theta) = V(\theta). \end{aligned} \tag{10.6}$$

As a consequence of (10.4) and (10.5) we have:

(10.7) *If one of* U, $V \equiv 0$, *the other is* $\not\equiv 0$.
(10.8) *If* $V(\theta_0) = 0$ *then* $U(\theta_0) \neq 0$.

We shall now distinguish several cases.

I. $U(\theta) \equiv 0$, hence $V(\theta) \not\equiv 0$. Then the paths are the concentric circles $r = r_0 =$ const., and we do not have asymptotic stability.

II. $V(\theta) \equiv 0$, hence $U(\theta) \not\equiv 0$. The paths are the rays $\theta = \theta_0 =$ const. That is the critical point is a node. To have asymptotic stability we must have $U(\theta) < 0$ for every θ, that is $xX_m + yY_m$ must be a definite negative form, and in particular m must be odd.

III. $V(\theta) \not\equiv 0$ *and* $V(\theta)$ *has real roots*. Referring to [3], Ch. X, the only admissible type of critical point in this case is a stable node. The stability is then asymptotic. Let θ_0 be a root of $V(\theta)$. Then the ray $\theta = \theta_0$ is a path. The geometry shows at once that a necessary and sufficient condition for stability (hence for asymptotic stability) is that on each such ray the moving point (r_0, θ_0) tend to the origin. This will happen if and only if $U(\theta) < 0$ on every solution of $V(\theta) = 0$; that is that $U_{m+1}(x, y) < 0$ whenever $V_{m+1}(x, y) = 0$.

IV. $V(\theta) \not\equiv 0$ and $V(\theta)$ has no real roots. This corresponds to a focus and this focus must be stable. As a ray starts from $\theta = 0$ until $\theta = 2\pi$, the variable point $M(r, \theta)$ corresponding to the solution must start from a certain M_0 on $\theta = 0$ and return to a position M_1 nearer than M_0 to the origin. The equation for $r(\theta)$ is

$$\frac{dr}{d\theta} = \frac{rU}{V}, \quad \frac{dr}{r} = \frac{Ud\theta}{V}$$

whose solution is

$$r = r_0 \exp \int_0^\theta \frac{Ud\theta}{V}.$$

We will have $r(2\pi) < r_0$ if and only if

$$\int_0^{2\pi} \frac{U(\theta)\, d\theta}{V(\theta)} < 0. \tag{10.9}$$

This is the necessary and sufficient condition for asymptotic stability in the case under consideration.

To sum up then, the necessary and sufficient conditions for asymptotic stability for $p = 2$ are:

$V_{m+1}(x, y) \equiv 0$; $U_{m+1}(x, y)$ *definite negative* (*hence* m *odd*).

$V_{m+1}(x, y) \not\equiv 0$ *and has real factors*; $U_{m+1}(x, y) < 0$ *wherever*

$V_{m+1}(x, y) = 0$ (*origin excepted*).

$V_{m+1}(x, y)$ *definite (positive or negative) hence m odd; the inequality* (10.9) *holds*.

11. *The case* $p = 3$. In a recent paper Coleman ([8]) has obtained a set of sufficient conditions for the asymptotic stability for $p = 3$. Let the cartesian coordinates be x_1, x_2, x_3 and the equations, in obvious notation,

$$\dot{x} = X_m(x) \tag{11.1}$$

where the components of X_m are real forms of degree $m > 1$. Let $r = \| x \|$, the Euclidean norm, and set $y = x/r$. Then (11.1) becomes with suitable time,

$$\begin{aligned} &\text{(a)} \quad \dot{r} = (y' \cdot X(y))r, \\ &\text{(b)} \quad \dot{y} = X(y) - (y' \cdot X(y))y. \end{aligned} \tag{11.2}$$

Coleman's conditions, which we merely state, are:

(a) $y' \cdot X(y) < 0$ *wherever* $X(y) - y'X(y)$ *is zero*;

(b) $\int_0^T y' \cdot X(y)\, dt < 0$ *for each periodic solution* $y(t)$ *of* (11.2b) *whose period is* T.

The whole argument of Coleman rests upon the detailed analysis of the system (11.2b) which is a system on the 2-sphere of radius unity.

Appendix I: The stability theorems of Liapunov

12. In order to make the paper as self-contained as possible we recall the theorems of Liapunov and Četaev (improvement of Liapunov) which have been applied in the text. They are not their full theorems but only the parts which refer to autonomous systems.

Take than an autonomous n-vector real analytical system

$$\dot{x} = X(x), \qquad X(0) = 0 \tag{12.1}$$

with the origin 0 as an isolated critical point. Assume that the system operates in a region Ω: $\| x \| < A$. A scalar function $V(x)$ will be called a Liapunov function over Ω whenever:

(a) $V(x)$ is of class C^1 in Ω;

(b) V is >0 in $\Omega - 0$ and $V(0) = 0$.

The time derivative $\dot{V}(x(t))$ along a solution $x(t)$ of (11.1) is given by $\dot{V} = \operatorname{grad} V \cdot X$.

Stability Theorem of Liapunov (12.2). *If there exists a Liapunov function* $V(x)$ *over* Ω *such that* $\dot{V} \leqq 0$ *then the origin is stable for* (11.1).

Asymptotic Stability Theorem of Liapunov (12.3). *If in addition* $-\dot{V}$ *is likewise a Liapunov function then the origin is asymptotically stable.*

Instability Theorem of Četaev (12.4). *Let* Ω_1 *be a subregion of* Ω *(it may be* Ω *itself) and let* B *be its boundary interior to* Ω. *Suppose that* B *contains the origin.*

Let there exist a scalar function $V(x)$ over Ω_1 + boundary, which is of class C^1 over that set, positive in Ω_1, zero on B. Let $\dot{V} > 0$ over Ω_1 together with B, and let $\dot{V}$ be such that given any $\alpha > 0$ there is a $\beta > 0$ such that whenever $V > \alpha$ in Ω_1 then $\dot{V} > \beta$. Under these conditions the origin is unstable.

In the applications in the text the regions Ω, Ω_1 are not explicitly described as they can be readily obtained from the text.

The case of linear equations with constant coefficients deserves special mention. Take

$$\dot{x} = Ax$$

where A is a constant stable $n \times n$ matrix. It is well known that when A is stable the origin is asymptotically stable. It has been proved by Liapunov that there exists then a *positive definite quadratic form* $V(x)$ such that $\dot{V}(x) = W(x)$ is a *negative definite quadratic form*. In other words V and $-A(x) = W(x)$ is a *negative definite quadratic form*. In other words V and $-\dot{V}$ may then be chosen as positive definite quadratic forms. In fact one may take for $\dot{V}$ any negative definite quadratic form and obtain from

$$\text{grad } V \cdot Ax = W$$

a unique solution for V as a positive definite quadratic form. The region Ω is here the whole space.

Appendix II: A theorem of Zubov

13. Consider the same system (12.1) where this time the components of X are homogeneous forms of the same degree $N > 1$,

$$\dot{x} = X(x) \tag{13.1}$$

We propose to prove the following special case of a proposition due to Zubov ([7], p. 189):

THEOREM (13.2). *If (13.1) is asymptotically stable there exist two Liapunov functions which are forms V, $-W$, where V is of degree two, W of degree $(N + 1)$ and $\dot{V} = W$. Hence N is odd and so $\geqq 3$.*

It is actually convenient in the calculations to write $N - 1 = \sigma$.

14. We must first prove two simple lemmas:

LEMMA (14.1). *If $x(t, x_0)$ is the solution such that $x(0, x_0) = x_0$, then whatever the scalar $c \neq 0$, $y(t) = cx(tc^\sigma, x_0)$ is the solution such that $y(0) = cx_0$.*

If we set $tc^\sigma = \tau$ then at once

$$\dot{y} = \frac{cdx(\tau, x_0)}{d\tau} \cdot \frac{d\tau}{dt} = c^{\delta+1} \frac{dx(\tau, x_0)}{d\tau} = c^N X(x) = X(y)$$

and $y(0) = cx_0$.

LEMMA (14.2). *If* (13.1) *is asymptotically stable there exist constants* a, $\tau > 0$ *such that any solution satisfies*:

$$\| x(t) \| \leqq a \cdot \| x(0) \| t^{-1/\sigma}, \qquad t \geqq \tau. \tag{14.2a}$$

Let $0 < k < 1$ and let $S(q)$ denote the sphere $\| x \| = k^q$.

Since the system is asymptotically stable every solution starting say at time $t = 0$ from a point of $S(0)$ will after a certain time T penetrate $S(1)$. Let τ be the supremum of the time that any such solution takes to reach $S(1)$. Because

$$\frac{d(kx)}{d(t/k^\sigma)} = X(kx) = k^N X(x)$$

the analogous time to jump from $S(1)$ to $S(2)$ is $\leqq k^{-\sigma}\tau$, etc. Hence the time for any trajectory initiating on $S(1)$ to reach $S(q)$ is

$$\leqq \tau(1 + k^{-\sigma} + k^{-2\sigma} + \cdots + k^{-\sigma(q-1)})$$
$$= \tau\left(\frac{k^{-\sigma q} - 1}{k^{-\sigma} - 1}\right) \leqq Ck^{-\sigma q} = t_1$$

where C is some positive constant. If M is any point on one of the paths between $S(q - 1)$ and $S(q)$ the time when it is reached is $t(M) < t_1$. Also

$$\|x(t)\| \leqq k^{q-1} \leqq C^k t_1^{-1/\sigma} \leqq C^k t^{-1/\sigma}.$$

From this together with (14.1) it is but a step to the lemma.

15. We are now ready for the proof of the theorem. Let $W(x) = \|x\|^{N+1}$. Thus $W(x)$ is a positive definite form of degree $N + 1$. Let $x(t, x_0, t_0)$ denote the solution of (13.1) such that $x(t_0, x_0, t_0) = x_0$. Since the system is autonomous we have $x(t, x_0, t_0) = x(t - t_0, x_0, 0)$. Write simply $x(t, x_0)$ for $x(t, x_0, 0)$.

Set now

$$V(x_0) = -\int_{t_0}^{+\infty} W(x(t, x_0, t_0))\, dt = \int_0^{+\infty} x(t, x_0)\, dt. \tag{15.1}$$

Since $N + 1 > 2$, in view of (14.2) one sees at once that the integral just written converges. Thus $V(x)$ is positive definite and analytic.

In the notations of lemma (14.1)

$$V(y_0) = V(cx_0) = -\int_0^{+\infty} W(cx(c^\sigma t, x_0))\, dt$$

$$= -c^{N+1}\int_0^{+\infty} W(x(\tau, x_0))c^{-\sigma}\, d\tau = -c^{N+1-\sigma}\int_0^{+\infty} W(x(\tau, x_0))\, d\tau = c^{N+1-\sigma}V(x_0).$$

Hence $V(x)$ is homogenous of degree $N + 1 - \sigma = 2$. Since it is analytic it is a quadratic form in x. Now as x_0 moves along the path of $x(t, x_0, t_0)$, this path does not change. Therefore,

$$\frac{d}{dt_0} W(t, x_0(t_0)t_0) = 0.$$

Hence now from (15.1)

$$\frac{dV(x_0(t_0))}{dt_0} = W(x(t_0, x_0, t_0)) = W(x_0)$$

or $\dot{V}(x(t)) = W$. This completes the proof of the theorem.

RIAS, Baltimore, Maryland, and
Universidad Nacional Autónoma de México

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