THE DEFINITION OF FIELD OF DEFINITION[1]

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1. Introduction

The fact that Weil did not see fit to define abstract algebraic sets in complete generality, restricting himself to such that are irreducible, has occasioned a number of well-known difficulties. For example, the algebraic set consisting of two curves intersecting in some complicated way should be definable without an embedding in the large, and one should be able to introduce the notion of field of definition for such a thing. On a very concrete level, it was known from the beginning that in the "correct" definition of "algebraic group defined over k" each component need not be defined over k, but the correct definition of field of definition is not easy to give in Weil's system. Thus the multiplicative group of the complex n^{th} roots of unity ought certainly to be defined over the rational numbers, but what should one say of a group that does not have such a nice affine embedding?

Of course all problems such as these can be answered by working in the vast generality of Grothendieck's system, but it is obviously worthwhile to see specifically what happens in the classical case of Weil's geometry, and that by using Weil's methods. This we do here, modifying some of his definitions (in some cases with slight change of terminology) and giving the results as they then are. In most cases the line of reasoning one would use to prove the new statement of a result is quite standard and entirely obvious, at least as long as not too many steps are skipped at once, so in the next two sections we confine ourselves to giving a more-or-less natural and reasonably complete sequence of definitions and results without proof. In line with our specific objectives, no mention is made of sheaves. Nor do we concern ourselves with historical matters, although mention should be made of the definition of "algebraic set" given in [1, pp. 1–12], which is roughly the same as ours.

A final section discusses the new definitions for algebraic groups and gives some simple applications.

2. Affine algebraic sets

Fix a universal domain Ω, an arbitrary algebraically closed field (of infinite transcendence degree over the prime field, if one wishes to use the technique of generic points). A set $V \subset \Omega^n$ is called *closed* if it is the zero-locus of some subset of $\Omega[X_1, \cdots, X_n]$, where $X_1, \cdots, X_n$ are indeterminates; this puts the Zariski topology on Ω^n, which is now compact. Call $V \subset \Omega^n$ *k-closed*, where k is some subfield of Ω, if V is the zero-locus of some subset of $k[X_1, \cdots, X_n]$ and call the complement of such a set *k-open*. The Hilbert Nullstellensatz says that if $V \subset \Omega^n$ is k-closed then $V \cap \bar{k}^n$ is dense in V, $\bar{k}$ being the algebraic closure of k. Any

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automorphism σ of Ω induces a homeomorphism of Ω^n, and if V is a closed subset of Ω^n and k a subfield of Ω then V is k-closed if and only if $V^\sigma = V$ for all k-automorphisms σ of Ω. If $V \subset \Omega^n$ is k-closed and $I(V)$ is the ideal consisting of all elements of $\Omega[X_1, \cdots, X_n]$ vanishing on V, then V is also the zero-locus of $I_k(V) = I(V) \cap k[X]$; the Ω-valued functions on V induced by elements of $\Omega[X_1, \cdots, X_n]$ form the *coordinate ring* $\Omega[V]$ of V, isomorphic to $\Omega[X]/I(V)$ and containing the subring $k[V]$ isomorphic to $k[X]/I_k(V)$. A subset $V \subset \Omega^n$ is called *rational over* k (instead of *defined over* k) if it is k-closed and $I(V) = \Omega[X] \cdot I_k(V)$; the subfields of Ω over which a given closed subset $V \subset \Omega^n$ is rational are all subfields containing a unique minimal such field k, which is a finite extension of the prime field with the property that an automorphism of Ω maps V into itself if and only if it leaves each element of k fixed. If $V \subset \Omega^n$ is rational over k then $k[V]$ and Ω (both subrings of $\Omega[V]$) are linearly disjoint over k, so $\Omega[V] = \Omega \otimes_k k[V]$; conversely if V is k-closed and $k[V]$ and $k^{p^{-\infty}}$ (= least perfect field containing k) are linearly disjoint over k, then V is rational over k. If the subset $V \cap k^n$ of the closed subset $V \subset \Omega^n$ is dense in V, then V is rational over k.

A k-closed subset $V \subset \Omega^n$ that is *irreducible over* k (that is, not the union of two k-closed proper subsets) is called a (*affine*) *k-variety*; the condition is that $I_k(V)$ be a prime ideal in $k[X]$, or that $k[V]$ be an integral domain. Any k-closed subset of Ω^n is an irredundant union of finitely many k-varieties (its *k-components*) in a unique manner; Ω-components are called, simply, *components*, and Ω-varieties are called *varieties*. If $V \subset \Omega^n$ is a k-variety and $V \cap k^n$ is dense in V, then V is a variety. The components of a k-variety V are rational over $\bar{k}$ and conjugate over k. A k-closed set $V \subset \Omega^n$ is rational over k if and only if each of its k-components is rational over k, which is true if and only if each component of V is rational over k_s (= separable part of the algebraic closure of k). A k-variety V is rational over k if and only if the field of quotients $k(V)$ of $k[V]$ is separable over k and it is a variety if and only if each element of $k(V)$ that is algebraic over k is purely inseparable over k.

Let V be a k-closed subset of Ω^n. The full ring of quotients $\Omega(V)$ of $\Omega[V]$ is the *ring of rational functions on* V, and the elements of $k(V)$, the full ring of quotients of $k[V]$, which is a subring of $\Omega(V)$, are the *rational functions over* k *on* V. If $V_1, \cdots, V_s$ are the k-components of V, then $k(V) = k(V_1) \oplus \cdots \oplus k(V_s)$ and each $k(V_i)$ is a field. For $P \in V$ define $\mathfrak{o}_k^V(P)$, the *semilocal ring of* P *in* $k(V)$, by

$$\mathfrak{o}_k^V(P) = \{u/v \mid u, v \in k[V], v \text{ not a zero-divisor in } k[V], v(P) \neq 0\},$$

and define $\mathfrak{o}^V(P)$, the *semilocal ring of* P *on* V, by $\mathfrak{o}^V(P) = \mathfrak{o}_\Omega^V(P)$. The elements of $\mathfrak{o}^V(P)$ are called the rational functions on V that are *defined at* P; in the special case that V is rational over k we have $k(V) \cap \mathfrak{o}^V(P) = \mathfrak{o}_k^V(P)$. If $V' \subset V$ is the union of certain k-components of V, including all those that contain P, then the "restriction" $k(V) \to k(V')$ induces a surjection $\mathfrak{o}_k^V(P) \to \mathfrak{o}_k^{V'}(P)$, and if V' is minimal then $\mathfrak{o}_k^{V'}(P)$ is a local ring and $\mathfrak{o}_k^V(P) = \mathfrak{o}_k^{V'}(P) \oplus k(V'')$, where V'' is the union of the remaining k-components of V. For a given $f \in k(V)$, the points $P \in V$ such that $f \in \mathfrak{o}_k^V(P)$ form a k-open dense subset of V, and $\bigcap_{P \in V} \mathfrak{o}_k^V(P) = k[V]$.

If V,W are k-closed subsets of Ω^n, Ω^m respectively, then $V \times W$ is k-closed in Ω^{n+m}, and in fact $I_k(V)$ and $I_k(W)$ together give a basis for $I_k(V \times W)$ if either V or W happens to be rational over k; as a consequence, if both V and W are rational over k, so is $V \times W$. If V,W are k-varieties one of which is irreducible, then $V \times W$ is a k-variety. By a *rational map over k from V into W* we mean a closed subset $\Phi \subset V \times W$ such that there exist elements $f_1, \cdots, f_m \in k(V)$ such that Φ is the closure of the set of all points $(P, f_1(P), \cdots, f_m(P))$, where P ranges over all points of V such that each $f_i \in \mathfrak{o}_k^V(P)$. If V is rational over k, so is the above rational map. The rational map Φ clearly determines $f_1, \cdots, f_m$, and conversely any $f_1, \cdots, f_m \in k(V)$ determine a rational map over k from V into Ω^m. The above rational map is said to be *defined at a point* $P \in V$ if each $f_i \in \mathfrak{o}^V(P)$, and is said to be *defined over k at P* if each $f_i \in \mathfrak{o}_k^V(P)$; the points of V at which the rational map is defined (resp. defined over k) are therefore dense k-open subsets of V, these two subsets coinciding if V is rational over k. We often identify Φ with the associated map φ from the subset of V on which the map is defined into W; φ is then continuous (where it is defined) and if $\varphi\{V\}$ denotes the closure of the image of φ then $\varphi\{V\}$ is k-closed, is a k-variety if V is, and is rational over k if V is.

3. Algebraic sets

By an *algebraic set over k* is meant a topological space V, together with a set of functions $\mathfrak{F} = \{f: D_f \to \Omega\}$ from certain dense open subsets $\{D_f\}$ of V into Ω such that

(1) if $\{U_\alpha\}_{\alpha \in A}$ is a set of open subsets of V and $f: \bigcup_{\alpha \in A} U_\alpha \to \Omega$ is a function whose restriction to each U_α is the restriction to U_α of some $f_\alpha \in \mathfrak{F}$, then f is the restriction to $\bigcup_{\alpha \in A} U_\alpha$ of some element of $\mathfrak{F}$; and such that

(2) there exists a finite set $\{W_i\}_{i \in I}$ of k-closed affine algebraic sets and maps $\varphi_i: W_i \to V$, $i \in I$, such that

(a) $V = \bigcup_{i \in I} \varphi_i(W_i)$,

(b) each $\varphi_i: W_i \to V$ is a homeomorphism onto an open subset of V,

(c) for each $i \in I, f \in \mathfrak{F}$, there is a rational function in $k(W_i)$ that is defined over k precisely on $\varphi_i^{-1}(D_f)$ and agrees there with $f\varphi_i$, and all elements of $k(W_i)$ arise in this way,

(d) if $i, j \in I$ then the set $\{(\varphi_i^{-1}(x), \varphi_j^{-1}(x)) \mid x \in \varphi_i(W_i) \cap \varphi_j(W_j)\}$ is closed on $W_i \times W_j$.

Thus an algebraic set over k is the union of a finite set of affine algebraic sets over k, with certain identifications. Indeed, $\varphi_j^{-1}\varphi_i$ defines a rational map over k from certain components of W_i into W_j, the map being defined over k precisely on $\varphi_i^{-1}(\varphi_j(W_j))$, with the inverse rational map $\varphi_i^{-1}\varphi_j$ defined on the image of this set; conversely a finite set of affine algebraic sets over k, $\{W_i\}_{i \in I}$, together with rational maps $\{\varphi_j^{-1}\varphi_i\}_{i,j \in I}$ from certain components of W_i into W_j, will give an algebraic set over k provided certain necessary conditions are satisfied. Defining an affine open subset over k of an algebraic set V over k in the obvious manner (as an affine algebraic set W over k together with a map $\varphi: W \to V$ satisfying (2) (b),(c)), one verifies that if we adjoin to the above set $\{W_i, \varphi_i\}$ any other affine open subset of V over k then condition (2)(d) is automatically satisfied, so that the above $\{W_i, \varphi_i\}_{i \in I}$ may be taken to be *any* finite set of affine open subsets over k that cover V. As usual, the intersection of two affine open subsets over k of V is also such a set. An algebraic set over k clearly gives rise to an algebraic set over any field K between k and Ω, what is involved being merely the addition of more functions. Most of the notions for affine algebraic sets over k generalize to arbitrary algebraic sets over k, for example the notions of rational map over k, k-morphism (i.e. rational map over k that is everywhere defined over k), irreducibility (over k or Ω), (k-) components, k-closed and k-open subsets of an algebraic set over k (themselves algebraic sets over k). If V is as above, the set $\mathfrak{F}$ can be made into a ring $k(V)$ in the obvious way, and $k(V) = k(V_1) \oplus \cdots \oplus k(V_s)$, $V_1, \cdots, V_s$ being the k-components of V. Call V *rational over* k if each of the W_i's above is rational over k; then all affine open subsets of V over k are rational over k. The algebraic set V over k is rational over k if and only if the same is true for all k-components of V or, equivalently, if and only if $k(V)$ and Ω (both subrings of $\Omega(V)$) are linearly disjoint over k. Clearly the disjoint union or product of two algebraic sets over k is an algebraic set over k, rational over k if the original algebraic sets are. Among the many other results that reduce directly to the affine case is that if V is an algebraic set rational over k, $k[V]$ the ring of everywhere defined rational functions on V over k, and K any field between k and Ω, then $K[V] = K \otimes_k k[V]$ (last theorem of Weil's *Foundations*). A related result is that if V,W are algebraic sets rational over k then $k[V \times W] = k[V] \otimes_k k[W]$.

The dimension of a variety V is the transcendence degree of $\Omega(V)$ over Ω and the dimension of any algebraic set is the maximum dimension of its components; as usual, each component of a k-variety has the same dimension equal to the transcendence degree of $k(V)$ over k. There is equally little difficulty extending to algebraic sets over k the notion of completeness (independent of base field) and local notions, such as normality and simplicity of a point of an algebraic set over k (relative to k, or absolute, i.e. relative to Ω); for results on k-normality and k-simplicity of a point the local results of Zariski for projective varieties can be carried over immediately to the general case, and similarly for the absolute results of Weil. Similarly for Weil's intersection theory.

4. Algebraic groups

By definition an *algebraic group over* k is an algebraic set G over k, together with a group structure on G such that the map $G \times G \to G$ given by $(g_1, g_2) \to g_1 g_2^{-1}$ is a k-morphism. An *algebraic group rational over* k is an algebraic group over k that is rational over k (as an algebraic set). As usual, if the algebraic group G is rational over k then so is the identity element e, and so is the component of the identity G_0. There is a similar definition of *transformation space* *over* k (or *rational over* k) for an algebraic group over k (or rational over k). A *homogeneous space* is a transformation space on which the group operates transitively.

It is a direct consequence of known results that in a separable rational homomorphism over k of algebraic groups rational over k, the kernel is rational over k; but what of the converse, i.e., how do we construct factor groups without extending the field of rationality? One method would be that of Weil, whose essence is the construction of honest-to-goodness algebraic groups and transformation spaces corresponding to similar objects that are given only "birationally" ([4]); this method can be abbreviated in part by applying known results to get this over a separable algebraic extension of the base field, at which point we have a field-descent problem that can be handled by [5] or [2, pp. 108–111]. In what follows we give a slight modification of a method of Chevalley ([1, exposé 8]).

The following Lemma 1 will be used repeatedly below. One immediate consequence of it will be that if $\tau: V \to V/G$ is a quotient space, with everything rational over k, then any rational function on V/G that is in $k(V)$ is actually in $k(V/G)$. Another consequence will be an immediate sharpening of the succeeding Lemma 3: if all the data in that lemma are rational over k, together with the relevant quotient spaces, then so is the isomorphism whose existence is affirmed.

LEMMA 1. *Let $V \to W \to Z$ be a sequence of rational maps of algebraic sets over k, with $V \to W$ and $V \to Z$ rational maps over k, the former inducing a morphism from its domain of definition onto a dense subset of W. Then if V is rational over k, so is the rational map $W \to Z$.*

We can assume that V, W are affine k-varieties and that Z is the affine line Ω, in which case the result says that any rational function $f \in \Omega(W) \cap k(V)$ is actually in $k(W)$. To see this, write $f = \Sigma a_i f_i / \Sigma a_i g_i$, where the sums are finite, each f_i, $g_i \in k[W]$, the denominator does not vanish on any component of W, and the various a_i's are elements of Ω that are linearly independent over k. The equation $\Sigma a_i(f_i - fg_i) = 0$ and the linear disjointness of Ω and $k(V)$ over k imply that $f_i - fg_i = 0$ for all i, giving $f \in k(W)$.

LEMMA 2. *If V is a homogeneous space for the algebraic group G, all rational over k, and $H \subset G$ is a normal algebraic subgroup, also rational over k, then there exists a quotient space V/H such that V/H and the natural projection $\tau: V \to V/H$ are rational over k.*

Recall that "$\tau: V \to V/H$ is a quotient space" means that τ is a morphism inducing a bijection between H-orbits on V and points of V/H such that any rational function on V that is constant on H-orbits and defined at any given point of V is actually a rational function on V/H that is defined at the corresponding point of V/H. The result is known to be true if k is algebraically closed, but it is worth mentioning that this can be proved directly in an easy way by first reducing to the case in which V is irreducible, then noting that a suitable k-open subset of any affine model over k of the field of H-invariant functions in

$k(V)$ is the quotient space modulo H of an H-invariant k-open subset of V, and finally obtaining V/H by using the transitivity of the operation of G on V and the unicity of quotient spaces. Having constructed $\tau: V \to V/H$ rational over $\bar{k}$, we have to show that V/H and τ can be modified in such a way as to make them both rational over k. In virtue of the unicity of quotient spaces, it suffices to prove that any $v \in V$ that is algebraic over k has a dense k-open H-invariant neighborhood V' such that V'/H exists and V'/H and the map $V' \to V'/H$ may both be chosen rational over k. To do this start with a dense $\bar{k}$-open affine subset $U \subset V/H$ and replace this by a suitable translate, if necessary, so as to be able to assume that τ maps all the k-conjugates of v into U. Letting σ range over the k-automorphisms of Ω, $\tau^\sigma: V \to (V/H)^\sigma$ is also a quotient space, hence isomorphic to $\tau: V \to V/H$, so $\tau^\sigma(\tau^{-1}(U))^\sigma$ is affine, hence also $\tau(\tau^{-1}(U))^\sigma$. Replacing U, if necessary, by the finite intersection $\bigcap_\sigma \tau(\tau^{-1}(U))^\sigma = \tau \bigcap_\sigma (\tau^{-1}(U))^\sigma$, we see that we could have chosen U such that $\tau^{-1}(U)$ is k-open on V. Thus any element of $\bar{k}[V]$ is of the form $\Sigma c_i f_i$, with the c_i's in $\bar{k}$ and linearly independent over k and each f_i in $k[\tau^{-1}(U)]$. By linear disjointness, each f_i is H-invariant, hence $\in \bar{k}[U]$. By using enough f_i's we get an isomorphic copy of U that is rational over k, proving Lemma 2.

LEMMA 3. *Let V_1, V_2 be homogeneous spaces for the algebraic groups G_1, G_2 respectively, and let H_1, H_2 be normal algebraic subgroups of G_1, G_2 respectively. Then $V_1 \times V_2$ is homogeneous for $G_1 \times G_2$ under the operation $(g_1, g_2)(v_1, v_2) = (g_1v_1, g_2v_2)$, and the equality $(H_1 \times H_2)(v_1, v_2) = (H_1v_1, H_2v_2)$ induces an isomorphism $(V_1 \times V_2)/(H_1 \times H_2) \approx (V_1/H_1) \times (V_2/H_2)$.*

The natural map $V_1 \times V_2 \to (V_1/H_1) \times (V_2/H_2)$ maps $H_1 \times H_2$-orbits into points, hence goes through $(V_1 \times V_2)/(H_1 \times H_2)$ (which exists by Lemma 2), i.e. we have a morphism $(V_1 \times V_2)/(H_1 \times H_2) \to (V_1/H_1) \times (V_2/H_2)$. This latter is bijective, and since it is also separable (component-wise) it is birational (component-wise), hence induces an isomorphism between certain dense open subsets. By the unicity of quotient spaces, each element of G_1 induces a translation on V_1/H_1 that is an isomorphism, and similarly for the operation of G_2 on V_2/H_2, and of $G_1 \times G_2$ on $(V_1 \times V_2)/(H_1 \times H_2)$. By transitivity, our isomorphism between dense open subsets is an isomorphism everywhere.

THEOREM 1. *Let H be an algebraic subgroup of the algebraic group G, both rational over k, and let H operate on G by the rule $(h,g) \to gh^{-1}$. Then the quotient space G/H (right coset space) exists, and G/H and the natural projecton $\tau: G \to G/H$ may be chosen rational over k, in which case G/H possesses the structure of a homogeneous space for G, also rational over k, such that $g_1(\tau g_2) = \tau(g_1g_2)$. If H is normal in G, then G/H possesses the structure of an algebraic group such that τ is a homomorphism, all rational over k. If, furthermore, V is a homogeneous space for G, also rational over k, then V/H exists and V/H and the natural projection $V \to V/H$ may be taken to be rational over k, in which case V/H possesses the structure of a homogeneous space for G/H, also rational over k, such that $(gH)(Hv) = Hgv$.*

H is normal in $G \times H$, which operates on G by the rule $(g,h)\gamma = g\gamma h^{-1}$, so, by Lemma 2, $\tau: G \to G/H$ exists, all rational over k. Applying Lemma 3 to $V_1 = G_1 = G$, $H_1 = \{e\}$, $V_2 = G$, $G_2 = G \times H$ (operating as above on $V_2 = G$), $H_2 = e \times H$, we get $(G \times G)/(e \times H) \approx G \times (G/H)$. But the k-morphism $G \times G \to G/H$ defined by $(g_1, g_2) \to g_1 g_2 H$ is $(e \times H)$-invariant, hence goes through $(G \times G)/(e \times H) \approx G \times (G/H)$, so we have a k-morphism $G \times (G/H) \to G/H$, giving the structure of G/H as a homogeneous space for G, rational over k. If H is normal in G, the above morphism $G \times G \to G/H$ is $(H \times H)$-invariant, hence goes through $(G/H) \times (G/H) \to G/H$, proving G/H an algebraic group rational over k, with τ a homomorphism. For the last part note that the k-morphism $G \times V \to V/H$ given by $(g, v) \to Hgv$ is $(H \times H)$-invariant, hence goes through $(G \times V)/(H \times H) \approx (G/H) \times (V/H)$; so V/H is indeed homogeneous for G/H, all rational over k.

THEOREM 2. *An algebraic group of matrices G that is rational over k is reducible to triangular form by a matrix rational over k if and only if G is solvable, $[G, G]$ consists of unipotent elements, and all rational characters of G are rational over k. In this case, if H is any algebraic subgroup of G that is rational over k and consists of semisimple elements, this can be done in such a way that H goes into a group of diagonal matrices.*

If G is reducible to triangular form by a matrix that is rational over k, then to prove our contentions we may suppose G in triangular form to begin with. Then G is clearly solvable and $[G, G]$ unipotent and it remains to show that any rational character of G is rational over k. The map τ which associates to each element of G the diagonal matrix with the same diagonal elements is a homomorphism rational over k whose kernel is the unipotent part G_u of G. We claim that τ is separable: it suffices to prove this for the component of the identity G_0 of G, so G can be assumed connected, and we may also suppose (extending the field k and conjugating G by a suitable triangular matrix if necessary) that G has a maximal torus T in diagonal form, in which case the separability of τ comes from the facts that τ is the identity on T and trivial on G_u and that $G = T \times G_u$. τ being separable and rational over k, all rational characters on G are rational characters on $G/G_u \approx \tau G$, a group of diagonal matrices, and it is known that for such a group all rational characters are rational over the prime field. Conversely, suppose that G satisfies the stated conditions. The same conditions then hold for any image of G under a rational homomorphism over k, so to prove that G can be put into triangular form by a matrix rational over k it suffices, by an obvious induction argument, to show that the vector space V on which G operates (supposed of dimension > 1) has a G-invariant subspace that is rational over k. If G is unipotent the subspace consisting of all $v \in V$ such that $gv = v$ for all $g \in G$ is known to be $\neq \{0\}$. This subspace, which is clearly k-closed, is also rational over k_s, since G has a dense subset of points rational over k_s and the subspace is given by linear equations. Therefore this subspace is rational over k, settling the case G unipotent. For G arbitrary, its subgroup

$[G, G]$ is normal, unipotent, and rational over k, and we have shown that the subspace $W \subset V$ consisting of all vectors on which $[G, G]$ operates trivially is rational over k; since W is G-invariant and $\neq \{0\}$, we may pass to the operation of $G/[G, G]$ on W to assume G commutative. Since G can now be triangulated (over Ω), there exists a rational character χ on G such that the set of all $v \in V$ such that $gv = \chi(g)v$ for all $g \in G$ is $\neq \{0\}$; the set of all these v's is a G-invariant subspace of V that is rational over k (by the same argument as above), so we are done, except for the contention about the subgroup H. But we already know that the vector space V on which G operates has a composition series $V = V_0 \supset V_1 \supset \cdots \supset V_n = \{0\}$, each V_i being a G-invariant subspace rational over k, and all we need do is pick, for each $i = 0, \cdots, n - 1$, an element of V_i, not in V_{i+1}, that is both rational over k and a characteristic vector for each element of H. Restricting our attention to V_i, we see that it suffices to prove that a group of semisimple matrices in triangular form that is rational over k can be diagonalized over k. But such a group is commutative, hence reducible to diagonal form over Ω, and a minor modification of the above argument proves diagonalizability over k.

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