ON THE COHOMOLOGY GROUPS OF THE CLASSIFYING SPACE FOR THE STABLE SPINOR GROUP

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1. Introduction

Denote by $SO(n) (n \geq 2)$ the group of $(n \times n)$ rotation matrices and let $Spin(n)$ denote the universal covering group of $SO(n)$. Thus we have the following exact sequence of groups,

$$1 \rightarrow Z_2 \rightarrow Spin(n) \rightarrow SO(n) \rightarrow 1,$$

where Z denotes the integers and Z_r the integers mod r $(r \geq 2)$. According to Borel, such an exact sequence of groups gives rise to a fiber space

$$B_{Z_2} \rightarrow B_{Spin(n)} \rightarrow B_{SO(n)} ,$$

where in general B_G denotes the classifying space for a topological group G. Now consider the stable groups ([14]),

$$SO = \bigcup_{n=2}^{\infty} SO(n), \quad Spin = \bigcup_{n=2}^{\infty} Spin(n).$$

Since the above fibering exists for each $n \geq 2$, we obtain in the limit the fiber space

$$K(Z_2, 1) \xrightarrow{\ i\ } \hat{B} \xrightarrow{\ \pi\ } B,$$

where $K(Z_2 , 1) = B_{Z_2} , \hat{B} = B_{Spin}$, and $B = B_{SO}$.

Because the integral cohomology groups of $K(Z_2 , 1)$ are either zero or Z_2 , it is clear that $\hat{B}$ and B have isomorphic cohomology groups with rational coefficients and with coefficients mod p, where p is an odd prime. The purpose of this paper is to compute the integral and mod 2 cohomology groups of $\hat{B}$.

Let us recall the results for B:

$$H^*(B;Z_2) = Z_2[W_2 , W_3 , \cdots]; H^*(B;Z) = Z[P_1 , P_2 , \cdots] \oplus T, \quad \text{where } 2T = 0.$$

Here $W_i \in H^i(B;Z_2)$ denotes the i^{th} Stiefel-Whitney class and $P_j \in H^{4j}(B;Z)$ denotes the j^{th} Pontrjagin class.[1] We shall show

Theorem (1.1). *For each positive integer i not of the form $2^r + 1$ $(r \geq 0)$, set $W_i^* = \pi^* W_i$. Then, $H^*(\hat{B};Z_2) = Z_2[W_4^*, W_6^*, W_7^*, \cdots]$.*

For any graded, anti-commutative algebra A denote by A_+ the ideal of positive dimensional elements. Define the *decomposition ideal* of A to be the ideal generated by $A_+ \cdot A_+$. In particular denote by D and D_2 the decompositions ideals

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[1] Strictly speaking W_i is characterized by the fact that when restricted to $H^*(B_{SO(n)}; Z_2) \cdot$ $(n \geq i)$, it becomes the i^{th} Stiefel-Whitney class of the classifying bundle over $B_{SO(n)}$—and similarly for P_j.

for the respective algebras $H^*(B;Z)$ and $H^*(B;Z_2)$. Set

$$D^i = D \cap H^i(B;Z), D_2^i = D_2 \cap H^i(B;Z_2) \quad (i \geq 0).$$

We shall prove

THEOREM (1.2). *There are cohomology classes* $\{Q_i\}$, $\{\Phi_i\}$, $\{\Psi_i\}$ $(i \geq 1)$ *with the following properties*:

(1.3) $Q_i \in H^{4i}(\hat{B},Z), \Phi_i \in D^{4i} \subset H^{4i}(B;Z), \quad \Psi_i \in D_2^i \subset H^i(B;Z_2).$

(1.4) *If i is not a power of 2, then*

$$Q_i = \pi^* P_i, \quad \Phi_i = 0, \quad \Psi_i = 0.$$

Let $j = 2^r$, *for* $r = 0, 1, \cdots$. *Then*

$$(1.5) \qquad \pi^* P_{2j} = 2Q_{2j} + Q_j^2 - \pi^*\Phi_{2j}, \qquad \pi^* P_1 = 2Q_1;$$

$$(1.6) \qquad \rho_2(Q_j) = \pi^*(W_{4j} + \Psi_{4j}), \qquad \rho_2(\Phi_j) = (\Psi_{2j})^2.$$

Moreover,

$$(1.7) \qquad H^*(\hat{B};Z) = Z[Q_1, Q_2, \cdots] \oplus \hat{T}, \quad \textit{where } 2\hat{T} = 0.$$

Furthermore, if $\{Q_i'\}$ $(i \geq 1)$ *is a second set of cohomology classes satisfying* (1.3)–(1.6) (*relative to a fixed choice of the* Φ's *and the* Ψ's) *then* $Q_i = Q_i'$.

Here $\rho_r (r \geq 2)$ denotes the cohomology homomorphism induced by the factor map $Z \rightarrow Z_r$. In Section 5 we give a specific choice for the classes Φ and Ψ. The low dimensional values of these classes are as follows:

$$(1.8) \qquad \begin{array}{ll} \Phi_i = 0 \quad \textit{for} \quad 1 \leq i \leq 7; & \Phi_8 = P_2P_6 + P_3P_5; \\ \Psi_i = 0 \quad \textit{for} \quad 1 \leq i \leq 15; & \Psi_{16} = W_4W_{12} + W_6W_{10}. \end{array}$$

Let η be an SO-bundle over a complex K and suppose that η is induced by a map f from K to B. Suppose that $W_2(\eta) = 0$. Then, as is well-known, f may be factored into a composition

$$K \xrightarrow{g} \hat{B} \xrightarrow{\pi} B,$$

where the map g is unique up to homotopy. Thus we call η a *Spin bundle*. Recent work of Atiyah, Borel, Hirzebruch and others (see, for example, [1], [3]) indicates the importance of these bundles. Taking the unique cohomology classes Q_i given in (1.2) (relative to the choice for the Φ's and Ψ's given in §5), define

$$(1.9) \qquad Q_i(\eta) = g^*Q_i \in H^{4i}(K;Z) \quad (i \geq 1)$$

and call these cohomology classes the *Spin characteristic classes* of the bundle η.

Let K denote the CW-complex which consists of a 1-sphere with a 2-cell attached by a map of degree 4. Denote by K_2 the 2-fold suspension of K. From the standpoint of K_2, the complex $\hat{B}$ may be considered an Eilenberg-MacLane space of type $(Z,4)$. Pick a generator u for the group $H^4(K_2;Z) \approx Z_4$. By (1.1), (1.5) and (1.6) one sees that the element Q_1 generates $H^4(\hat{B};Z) \approx Z$. Thus there is a map g from K_2 to $\hat{B}$ such that $g^*Q_1 = 2u$. Denote by η the *Spin*-bundle over K_2 corresponding to the map $\pi \circ g$. Then,

$$P_1(\eta) = (\pi \circ g)^* P_1 = g^*(2Q_1) = 4u = 0;$$
$$W_4(\eta) = (\pi \circ g)^* W_4 = g^*(\rho_2 Q_1) = \rho_2(2u) = 0;$$
$$Q_1(\eta) = g^* Q_1 = 2u \neq 0.$$

Thus η is a bundle whose regular characteristic classes are trivial but for which $Q_1(\eta) \neq 0$. Using the Bott divisibility theorems one can construct an analogous complex K_{4k-2} and *Spin* bundle η_{4k-2} $(k \geq 1)$ such that

$$P(\eta_{4k-2}) = 1; \; W(\eta_{4k-2}) = 1; \; Q_k(\eta_{4k-2}) \neq 0.$$

While the classes Q_i are given uniquely by making a specific choice of the Φ's and the Ψ's, it would be more desirable to obtain the uniqueness by means of axioms which have geometric significance. As yet I have not been able to do this. In the axioms given for the regular characteristic classes (see [5], [6]) the behavior of the classes on the Whitney sum of two bundles has played an important role. Let η and ζ be two *Spin* bundles over the same base space. Then the Whitney sum $\eta \oplus \zeta$ is also a *Spin* bundle. Using (1.5), (1.6) and the known facts about Pontrjagin classes ([12, (3)]) one can easily show:

(1.10) *If* $k \leq 3$, *then* $Q_k(\eta \oplus \zeta) = \sum_{i+j=k} Q_i(\eta) \smile Q_j(\zeta)$.

$$Q_4(\eta \oplus \zeta) = \sum_{i+j=4} Q_i(\eta) \smile Q_j(\zeta) + (Q_2(\eta) + Q_2(\zeta))(Q_1(\eta) \smile Q_1(\zeta)).$$

The formula for the higher dimensional Q's will be very complicated, and I do not know it explicitly.

The proof of (1.1) is given in the following section while the subsequent sections are devoted to the proof of (1.2). I would like to thank John Milnor for his helpful suggestions concerning this paper.

2. Proof of (1.1)

Let X be a topological space that has finitely-generated (integral) cohomology groups in each dimension. Taking cohomology groups with coefficients in a field k, we will say that $H^*(X)$ has a simple system of generators (see Borel, [2]) if there are elements $x_0, x_1, x_2, \cdots \in H^*(X)$ such that the totality of monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$ $(0 \leq i_1 < i_2 < \cdots < i_r, r \geq 1)$ forms a k-vector space basis for $H^*(X)$. In this case we write $H^*(X) = \Delta(x_0, x_1, \cdots)$.

Consider now a (Serre) fiber space $F \xrightarrow{i} E \xrightarrow{\pi} B$, where we assume for simplicity that the base space B is 1-connected. Recall that one then has the transgression operator, which is a homomorphism of degree $+1$ from a subgroup of $H^*(F)$ to a factor group of $H^*(B)$.

LEMMA (2.1). *Taking coefficients in a field* k, *suppose that*

$$H^*(F) = \Delta(x_0, x_1, \cdots), \qquad H^*(B) = k[y_0, y_1, \cdots] \otimes Q,$$

where x_i is transgressive ($i \geq 0$), *y_i represents its transgression and Q is a subalgebra of $H^*(B)$. If* char $k \neq 2$, *assume each x_i has odd dimension. Then,* $\pi^*: Q \approx H^*(E)$, *where π^* denotes the cohomology homomorphism induced by π.*

The proof follows very easily from Theorems (13.1), (16.1) of [2] and the spectral sequence comparison theorem of Zeeman ([15]). We leave the details to the reader.

We apply this[2] to the fiber space $K(Z_2, 1) \to \hat{B} \to B$, given in §1. Taking coefficients mod 2, let x_0 denote a generator for $H^1(Z_2, 1)$. Then $H^*(Z_2, 1) = Z_2[x_0]$. For $r \geq 1$, define

$$x_r = (x_0)^{2^r} = \mathrm{Sq}^{2^{r-1}} \circ \cdots \circ \mathrm{Sq}^2 \circ \mathrm{Sq}^1(x_0),$$

where Sq^i denotes the Steenrod operator. Thus,

$$H^*(Z_2, 1) = \Delta(x_0, x_1, \cdots). \tag{2.2}$$

Consider now the polynomial algebra $H^*(B) = Z_2[W_2, W_3, \cdots]$, and recall the formula of Wu:

$$\mathrm{Sq}^i W_{i+1} = W_{2i+1} + \textstyle\sum_{2 \leq t \leq i} W_t W_{2i+1-t} \quad (i \geq 1). \tag{2.3}$$

Set

$$y_0 = W_2,\ y_r = \mathrm{Sq}^{2^{r-1}} \circ \cdots \circ \mathrm{Sq}^2 \circ \mathrm{Sq}^1 W_2 \quad (r \geq 1).$$

Then by (2.3) and the Cartan product formula,

$$y_r \equiv W_{2^r+1} \mod D_2$$

and therefore,

$$\begin{aligned} H^*(B) &= Z_2[y_0, y_1, W_4, y_2, W_6, W_7, \cdots] \\ &= Z_2[y_0, y_1, y_2, \cdots] \otimes Z_2[W_4, W_6, W_7, \cdots]. \end{aligned} \tag{2.4}$$

Now the element $x_0 \in H^*(Z_2, 1)$ is transgressive (in the fibering we are considering) and y_0 is its (unique) representative. Since the transgression operator commutes with the Steenrod operations, we see that each element x_i ($i \geq 1$) is transgressive and y_i represents its transgression. Therefore, (1.1) follows at once from (2.1), (2.2), and (2.4).

3. The Pontrjagin square

Recall that $H^*(B;Z) = Z[P_1, P_2, \cdots] \oplus T$. Denote by L the subset of $Z[P_1, P_2, \cdots]$ which consists of those polynomials whose non-zero coefficients are all $+1$, together with the zero polynomial. Set $M = L + T \subset H^*(B;Z)$; that is, a cohomology class $u \in M$ if, and only if, $u = l + t$, where $l \in L$, $t \in T$. The following facts are then easily ascertained.

[2] The following proof is based on the one given by Borel for Propositions (15, 2) and (15, 21) in "*Sur l'homologie et la cohomologie des groupes de Lie compact connexes*, Amer. Jr of Math., **76** (1954), pp. 273–342.

(3.1) *Let x and y be monomials in L. Then xy is a monomial in L.*

(3.2) *Let $z \in M$ and $t \in T$. Then $tz = zt \in T \subset M$. If* $\dim z = \dim t$, *then* $z + t \in M$.

(3.3) $\rho_2 \mid M$ *is a monomorphism.*

We show

LEMMA (3.4). *Let $u \in H^*(B;Z_2)$. Then there is a unique element $U \in M$ such that $\rho_2(U) = u^2$. Moreover, if u is decomposable, then so is U.*

The uniqueness of the class U follows from (3.3). To show the existence suppose first that u is a generator for the polynomial algebra $H^*(B;Z_2)$. Since

$$\rho_2(P_i) = (W_{2i})^2, \quad \rho_2(\delta \mathrm{Sq}^{2i} W_{2i+1}) = (W_{2i+1})^2,$$

and since $P_i \in L$, $\delta \mathrm{Sq}^{2i} W_{2i+1} \in T$, we have proved (3.4) for this case. (Here δ is the Bockstein coboundary from Z_2 to Z.) Suppose next that u is a monomial of degree > 1. Then (3.4) follows at once from (3.1), (3.2) and the fact that ρ_2 is a multiplicative homomorphism. Moreover, when u is a monomial, the class U is then either a monomial in L or an element of T. Now let $u = u_1 + \cdots + u_r$, where the u_i's are distinct monomials in $H^*(B; Z_2)$. Then there are elements $U_1, \cdots, U_r \in M$ such that $\rho_2(U_i) = u_i^2$, and we may number these elements in such a way that for some integer $q \leqq r$, the elements $U_1, \cdots, U_q$ are monomials in L and $U_{q+1}, \cdots, U_r \in T$. Since $u_1, \cdots, u_q$ are all distinct monomials it is clear that $U_1 + \cdots + U_q$ is a polynomial in L. Hence, setting $U = (U_1 + \cdots + U_q) + (U_{q+1} + \cdots + U_r)$ we obtain the desired class, recalling that

$$u^2 = (u_1 + \cdots + u_r)^2 = u_1^2 + \cdots + u_r^2.$$

For any class $u \in H^*(B; Z_2)$ we define its *integral representative* to be the unique class $U \in M$ given in (3.4). Thus, if $\dim u = q$, then $\dim U = 2q$, and $\rho_2(U) = u^2$. Also, if u is decomposable, then so is U.

Define P to be the subalgebra of $H^*(B; Z_2)$ which is generated by $W_2^2, W_4^2, W_6^2, \cdots, W_{2i}^2, \cdots$. Define S to be the (vector) subspace of $H^*(B; Z_2)$ which has as basis the totality of monomials

$$pW_{2i_1} \cdots W_{2i_a} W_{2j_1+1} \cdots W_{2j_b+1},$$

where $p \in P$, $i_1 < \cdots < i_a$, $a \geq 1$; $j_1 \leq \cdots \leq j_b$, $b \geq 0$; and $i_1 \leq j_1$, if $b > 0$. Finally denote by $\mathfrak{P}_2$ the Pontrjagin square and by θ the cohomology homomorphism induced by the inclusion $Z_2 \subset Z_4$. We show

LEMMA (3.5). *Let u be a monomial in $H^*(B; Z_2)$. Then there is a unique class $v \in S$ such that*

$$\mathfrak{P}_2(u) = \rho_4(U) + \theta(v),$$

where U is the integral representative for u. Moreover, if u is decomposable then so is v.

Recall from [11, §2] that we have a vector space splitting

$$H^*(B; Z_2) = P \oplus \beta S \oplus S, \tag{3.6}$$

where $P \oplus \beta S = \text{Kernel } \beta$, $\beta S = \text{Image } \beta$. Here β is the Bockstein coboundary $\rho_2\delta$. Since Kernel $\theta = \text{Image } \beta = \beta S$, it is clear that $\theta \mid S$ is a monomorphism and hence the class v is unique.

Now Wu [14], [10] has shown that

$$(3.7) \quad \begin{aligned} \mathfrak{P}_2(W_{2j}) &= \rho_4(P_j) + \theta[W_{4j} + \textstyle\sum_{0<i<j} W_{2i}W_{4j-2i}], \\ \mathfrak{P}_2(W_{2j+1}) &= \rho_4(\delta \mathrm{Sq}^{2j} W_{2j+1}), \quad \text{for} \quad j \geq 1. \end{aligned}$$

Define

$$\Omega_{4j} = \textstyle\sum_{0<i<j} W_{2i}W_{4j-2i} \quad (j \geq 1),$$

which clearly is an element in S. Suppose now that the monomial u, in (3.5), is a generator $W_k (k \geq 2)$. Setting

$$\begin{aligned} U &= P_j, \qquad v = W_{4j} + \Omega_{4j} \quad (\text{if } k = 2j) \\ U &= \delta \mathrm{Sq}^{2j} W_{2j+1}, \qquad v = 0 \quad (\text{if } k = 2j + 1), \end{aligned}$$

we obtain the desired classes.

We complete the proof of (3.5) by an inductive argument. Suppose that the lemma has been proved for monomials of degree $n (n \geq 1)$, and let u be a monomial of degree $n + 1$. Then, $u = u_1W_i$, where u_1 is a monomial of degree n and $i \geq 2$. Therefore there are classes $U_1, \bar{U} \in M$, and $v_1, \bar{v} \in S$ such that

$$\mathfrak{P}_2(u_1) = \rho_4(U_1) + \theta(v_1), \qquad \mathfrak{P}_2(W_i) = \rho_4(\bar{U}) + \theta(\bar{v}).$$

Now by equation 4.5 (2) of [9],

$$\mathfrak{P}_2(u) = \mathfrak{P}_2(u_1)\mathfrak{P}_2(W_i) + \theta[(\mathrm{Sq}^{r-1}u_1)(W_i\beta W_i) + (u_1\beta u_1)(\mathrm{Sq}^{i-1}W_i)],$$

where $\dim u_1 = r > 0$. Consequently,

$$\mathfrak{P}_2(u) = \rho_4(U_1\bar{U}) + \theta[u_1^2\bar{v} + v_1W_i^2 + (\mathrm{Sq}^{r-1}u_1)(W_i\beta W_i) + (u_1\beta u_1)(\mathrm{Sq}^{i-1}W_i)].$$

Here we have used the fact that $\theta(v_1)\theta(\bar{v}) = 0$; that $\theta(a)\rho_4(b) = \theta(a\rho_2 b)$ for any classes $a \in H^*(B; Z_2)$, $b \in H^*(B; Z)$; and that $U_1, \bar{U}$ are the respective integral representatives for u_1 and W_i.

Denote by I the ideal of $H^*(B; Z_2)$ generated by the elements $W_3, W_5, \cdots, W_{2i+1}, \cdots$. Then the vector space $\beta S \oplus S$ is a module over both I and P. By (2.3)

$$\mathrm{Sq}^{i-1}W_i \in I, \quad W_i\beta W_i \in I;$$

and $(W_i)^2$ belongs either to I or P. But it is clear that no monomial term in $u_1^2\bar{v}$ belongs to P, since $\bar{v} \notin P$, and hence $u_1^2\bar{v} \in \beta S \oplus S$. Therefore, there are unique classes $x \in \beta S$, $v \in S$ such that

$$u_1^2\bar{v} + v_1W_i^2 + (\mathrm{Sq}^{r-1}u_1)(W_i\beta W_i) + (u_1\beta u_1)(\mathrm{Sq}^{i-1}W_i) = x + v. \tag{3.8}$$

Finally, the element $\bar{U}$ is either a monomial in L or an element in T; and by induction, the same is true of U_1. Thus, $U_1\bar{U} \in M$, by (3.1) and (3.2). Setting $U = U_1\bar{U}$ we obtain, since $\theta(x) = 0$,

$$\mathfrak{P}_2(u) = \rho_4(U) + \theta(v),$$

completing the inductive step.

We are left with showing that the class v is decomposable. Suppose that $z_1 \in \beta S \oplus S$ is a decomposable monomial (i.e., z_1 has degree > 1), and write $z_1 = x_1 + y_1$ where $x_1 \in \beta S$, $y_1 \in S$. It follows fairly easily from page 411 of [11] that x_1 and y_1 are also decomposable. Since the left hand side of (3.8) is decomposable, this shows that the class v is too, which completes the proof of the lemma.

We use (3.5) to obtain the main result of the section.

LEMMA (3.9). *Let u be any element in $S \subset H^*(B; Z_2)$. Then there is a unique element $v \in S$ such that*

$$\mathfrak{P}_2(u) = \rho_4(U) + \theta(v),$$

where U is the integral representative for u. Moreover, if u is decomposable then so is v.

The uniqueness of v follows, as before, from (3.3). Write $u = u_1 + \cdots + u_r$, where the u_i's are distinct monomials in S. By (3.5) there are classes $v_i \in S$ such that

$$\mathfrak{P}_2(u_i) = \rho_4(U_i) + \theta(v_i),$$

where U_i is the integral representative for u_i. Then $U = U_1 + \cdots + U_r$ is the integral representative for u. Furthermore, if u is decomposable then so is each monomial u_i—as is, by (3.5), each element v_i.

If u is odd dimensional, then by (7.7) of [8], $\mathfrak{P}_2$ is an additive operation, and

$$\mathfrak{P}_2(u) = \rho_4(U) + \theta(v),$$

where $v = v_1 + \cdots + v_r \in S$, completing the proof for this case. Suppose then that u is even dimensional. Then

$$\begin{aligned}\mathfrak{P}_2(u) &= \textstyle\sum_i \mathfrak{P}_2(u_i) + \sum_{i<j} \theta(u_iu_j) \\ &= \rho_4(U) + \theta(v) + \textstyle\sum_{i<j} \theta(u_iu_j).\end{aligned}$$

Since the u_i's are all distinct monomials in S, it follows from the definition of S that $u_iu_j \in S (i \neq j)$. Hence

$$\mathfrak{P}_2(u) = \rho_4(U) + \theta(\bar{v}),$$

where $\bar{v} = v + \sum_{i<j} u_iu_j \in S$. This completes the proof of the lemma.

4. Polynomial subrings

Consider a graded anti-commutative ring A, which is finitely generated in each dimension. Denote the rational numbers by R_0.

LEMMA (4.1). *Let $u_1, u_2, \cdots \in A$ be even dimensional elements and denote by U the subring of A generated by the u's. Suppose that*

(1) *$A \otimes R_0$ is a polynomial ring on $u_1 \otimes 1, u_2 \otimes 1, \cdots$;*

(2) *$U \otimes Z_p$ is a polynomial ring on $u_1 \otimes 1_p, u_2 \otimes 1_p, \cdots$, where $1_p = 1 \bmod p$ (all primes p);*

(3) *$(U \otimes Z_p) \cap (T \otimes Z_p) = 0$ (all primes p), where T denotes the torsion ideal of A.*

Then, U is a polynomial subring of A with $u_1, u_2, \cdots$ as generators, and $A = U \oplus T$ (group direct sum).

It is clear from (1) that U is a polynomial ring on $u_1, u_2, \cdots$; and hence $U \cap T = 0$. Thus we need simply show that every element $a \in A$ can be written as $u + t$, where $u \in U$ and $t \in T$.

Consider the exact sequence

$$0 \to T \xrightarrow{i} A \xrightarrow{\rho} A \otimes R_0,$$

where i is the inclusion and ρ is the ring homomorphism given by $\rho(a) = a \otimes 1$, for $a \in A$. Using this exact sequence together with (2) and (3), one may now complete the proof of (4.1) by exactly the same argument as that used to prove (7.1) in [10]. We leave the details to the reader.

There are several applications one can make of (4.1), but the following is the one needed for the proof of (1.7). Let X be a topological space whose integral cohomology groups are finitely generated in each dimension. Let $u_1, u_2, \cdots$ be even dimensional elements in $H^*(X)$ (integral coefficients) and let U be the subring generated by the u's. Denote by ρ_0 the cohomology homomorphism induced by the inclusion $Z \subset R_0$. We show

THEOREM (4.2). *Suppose that the cohomology groups of X have the following properties.*

(1) *$H^*(X; R_0) = R_0[\rho_0(u_1), \rho_0(u_2), \cdots]$;*

(2) *$H^*(X)$ has no p-torsion for odd primes p;*

(3) *$H^*(X; Z_2) = Z_2[x_1, x_2, \cdots; y_1, y_2, \cdots; z_1, z_2, \cdots]$, where $\beta x_i = y_i$, $\beta z_j = 0$;*

(4) *$\rho_2(U)$ is a polynomial ring with $\rho_2(u_1), \rho_2(u_2), \cdots$, as generators and $\rho_2(U) = Z_2[x_1^2, x_2^2, \cdots; z_1, z_2, \cdots]$.*

Then,

$$H^*(X) = Z[u_1, u_2, \cdots] \oplus T, \text{ where } 2T = 0.$$

Let T denote the torsion ideal of $H^*(X)$. We first show that $2T = 0$. In view of (2) this will be true if, and only if, Kernel β = Image ρ_2. Since one always has Image $\rho_2 \subset$ Kernel β, we need simply show that Kernel $\beta \subset$ Image ρ_2.

Recall that the coboundary β is a derivation (i.e., $\beta(uv) = (\beta u)v + u(\beta v)$). Therefore by property (3) above and Theorem 1 of [11], there is a (vector) subspace $S \subset H^*(X; Z_2)$ such that[3]

$$H^*(X; Z_2) = P \oplus \beta S \oplus S,$$

where $P = Z_2[x_1^2, x_2^2, \cdots, z_1, z_2, \cdots]$, and where

$$\text{Kernel } \beta = P \oplus \beta S, \qquad \text{Image } \beta = \beta S.$$

Now β is the composition $\rho_2\delta$, and therefore $\beta S = \rho_2\delta S \subset \text{Image } \rho_2$. By property (4), $P = \rho_2(U) \subset \text{Image } \rho_2$, and consequently, Kernel $\beta = P \oplus \beta S \subset \text{Image } \rho_2$, completing the proof that $2T = 0$.

Now by (6.5) of [10], $\rho_2(T) = \text{Image } \beta = \beta S$, and hence

$$\rho_2(U) \cap \rho_2(T) = 0. \tag{4.3}$$

Setting $A = H^*(X)$ (integral coefficients) we will apply (4.1) to obtain (4.2). Since

$$A \otimes R_0 = H^*(X) \otimes R_0 = H^*(X; R_0),$$

(4.2)(1) implies (4.1)(1). Moreover, as was remarked in the proof of (4.1), this already shows that U is a polynomial ring on $u_1, u_2, \cdots$. Now for any subset $V \subset H^*(X)$ and any prime p, $\rho_p(V) \approx V \otimes Z_p$. (This is true for any space X). Therefore, $\rho_p(U) = U \otimes Z_p$, and hence condition (4.1)(2) is fulfilled—using (4.2)(2) if p is odd, and (4.2)(4) if $p = 2$. Since $\rho_p(T) = 0$ if p is odd, condition (4.1)(3) is satisfied by (4.3). Therefore, the conclusion of (4.2) follows from the conclusion of (4.1).

In the next section we apply (4.2) to prove (1.7). Other applications of (4.2) can be made to the integral cohomology rings

$$H^*(B_{O(n)}), \qquad H^*(B_{SO(n)})$$

($2 \leq n \leq \infty$), where $O(n)$ denotes the group of orthogonal $(n \times n)$-matrices. (See Theorem A and (12.1) of [10].) Finally, an algebraic analogue of (4.2) can be used to give the structure of the Thom ring ([7])Ω of orientable manifolds (see [13] and [11; §2]).

5. Proof of (1.2)

Suppose we have defined the elements Ψ_i that occur in (1.2). We then define the Φ's by taking $\Phi_j (j \geq 1)$ to be the integral representative of Ψ_{2j}. Thus, dim $\Phi_j = 4j$, $\rho_2(\Phi_j) = \Psi_{2j}^2$, and $\Phi_j \in D$, since, by hypothesis, $\Psi_{2j} \in D_2$. By (3.4) the Φ's are given uniquely, once we specify the Ψ's. In order to define the

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[3] We are taking $k = Z_2[z_1, z_2, \cdots]$ in [11], and using the identification $H^*(X; Z_2) = k[x_1, x_2, \cdots; y_1, y_2, \cdots]$.

Ψ's canonically we will assume that

$$\Psi_i \in D_2^i \cap S \subset H^i(B; Z_2) \quad (i \geq 1),$$

where S is the subspace defined in §3.

If i is not a power of 2, set $\Psi_i = 0$, agreeing with (1.4). For i a power of two, we give a recursive definition for Ψ_i, beginning with $\Psi_1 = \Psi_2 = 0$. Suppose then that Ψ_k has been defined, where k is a power of $2 \geq 2$, and $\Psi_k \in D_2^k \cap S \subset H^k(B; Z_2)$. By (3.9) there is a unique class $\Lambda_{2k} \in D_2^{2k} \cap S \subset H^{2k}(B; Z_2)$ such that

$$\mathfrak{P}_2(\Psi_k) = \rho_4(\Phi_l) + \theta(\Lambda_{2k}), \tag{5.1}$$

where $l = k/2$. We define

$$\Psi_{2k} = \Lambda_{2k} + W_k\Psi_k + \hat{\Omega}_{2k}, \tag{5.2}$$

where $\hat{\Omega}_{2k} = \sum_{1<i<l} W_{2i}W_{2k-2i}$. Clearly $\Psi_{2k} \in D_2^{2k} \cap S$. Since $\hat{\Omega}_2 = \hat{\Omega}_4 = \hat{\Omega}_8 = 0$, the first non-zero value of Ψ_{2k} is

$$\Psi_{16} = \hat{\Omega}_{16} = W_4W_{12} + W_6W_{10}.$$

Since this is a decomposable class in S_{16}, the recursion starts correctly.

Thus the Φ and Ψ-classes have been defined in (1.2), and agree with (1.8). Recall that $\rho_2(P_i) = (W_{2i})^2$ for $i \geq 1$. Therefore, $\rho_2(\pi^*P_1) = (\pi^*W_2)^2 = 0$, and consequently there is an element $Q_1 \in H^4(\hat{B}; Z)$ such that $\pi^*P_1 = 2Q_1$. We show that $\rho_2(Q_1) = \pi^*W_4$. Now by (3.7),

$$0 = \mathfrak{P}_2(0) = \mathfrak{P}_2(\pi^*W_2) = \rho_4(\pi^*P_1) + \theta(\pi^*W_4).$$

Since $2\rho_4 = \theta\rho_2$, we obtain

$$\theta(\rho_2Q_1 - \pi^*W_4) = 0$$

and hence $\rho_2Q_1 = \pi^*W_4$, since $H^3(\hat{B}; Z_2) = 0$. Therefore, the element Q_1 satisfies (1.3), (1.5) and (1.6). In a similar fashion one obtains an element Q_2 satisfying (1.3)–(1.6). For integers i that are not a power of 2 we take (1.4) as definition, and obtain the remaining Q's by an induction argument on powers of 2.

Suppose then that classes Q_j, satisfying (1.3)–(1.6), have been defined for all integers j which are powers of $2 \leq 2^r (r \geq 1)$. Set $l = 2^{r-1}$, $k = 2l = 2^r$. Thus we assume that we have a class $Q_k \in H^{4k}(\hat{B}; Z)$ such that

$$\begin{aligned} \pi^*P_k &= 2Q_k + Q_l^2 - \pi^*\Phi_k, \\ \rho_2Q_k &= \pi^*(W_{4k} + \Psi_{4k}). \end{aligned} \tag{5.3}$$

Now

$$\begin{aligned} \rho_2(\pi^*P_{2k}) &= (\pi^*W_{4k})^2 = (\rho_2Q_k - \pi^*\Psi_{4k})^2 \\ &= \rho_2Q_k^2 + \pi^*\Psi_{4k}^2 = \rho_2(Q_k^2 - \pi^*\Phi_{2k}), \end{aligned}$$

and therefore,

$$\rho_2(\pi^* P_{2k} - Q_k^2 + \pi^* \Phi_{2k}) = 0.$$

Consequently, by exactness, there is an element $\bar{Q}_{2k} \in H^{8k}(\hat{B}; Z)$ such that

$$\pi^* P_{2k} = 2\bar{Q}_{2k} + Q_k^2 - \pi^* \Phi_{2k}\,. \tag{5.4}$$

Using (3.7) and (5.4) we obtain[4]

$$\begin{aligned} \mathfrak{P}_2(\pi^* W_{4k}) &= \rho_4(\pi^* P_{2k}) + \theta(\pi^* W_{8k}) + \theta(\pi^* \hat{\Omega}_{8k}) \\ &= \rho_4(Q_k^2 - \pi^* \Phi_{2k}) + \theta[\rho_2 \bar{Q}_{2k} + \pi^* W_{8k} + \pi^* \hat{\Omega}_{8k}]. \end{aligned} \tag{5.5}$$

On the other hand, by (5.3) we have

$$\begin{aligned} \mathfrak{P}_2(\pi^* W_{4k}) &= \mathfrak{P}_2(\rho_2 Q_k + \pi^* \Psi_{4k}) \\ &= \mathfrak{P}_2(\rho_2 Q_k) + \mathfrak{P}_2(\pi^* \Psi_{4k}) + \theta(\rho_2 Q_k)(\pi^* \Psi_{4k}). \end{aligned}$$

By Theorem I (9) and Theorem II (ii) of [8], $\mathfrak{P}_2(\rho_2 Q_k) = \rho_4(Q_k^2)$, and by (5.1),

$$\mathfrak{P}_2(\pi^* \Psi_{4k}) = \rho_4(\pi^* \Phi_{2k}) + \theta(\pi^* \Lambda_{8k}).$$

Furthermore,

$$\begin{aligned} \theta(\rho_2 Q_k)(\pi^* \Psi_{4k}) &= \theta[\pi^*(W_{4k} \Psi_{4k} + \Psi_{4k}^2)] \\ &= \theta[\pi^*(W_{4k} \Psi_{4k} + \rho_2 \Phi_{2k})] \\ &= \theta(\pi^* W_{4k} \Psi_{4k}) + 2\rho_4(\pi^* \Phi_{2k}). \end{aligned}$$

Therefore,

$$\mathfrak{P}_2(\pi^* W_{4k}) = \rho_4(Q_k^2 + 3\pi^* \Phi_{2k}) + \theta \pi^*(\Lambda_{8k} + W_{4k} \Psi_{4k}). \tag{5.6}$$

Comparing (5.5) and (5.6) we obtain by (5.2),

$$\theta[\rho_2 \bar{Q}_{2k} + \pi^* W_{8k} + \pi^* \Psi_{8k}] = 0,$$

and hence by exactness,

$$\rho_2 \bar{Q}_{2k} + \pi^*(W_{8k} + \Psi_{8k}) = \beta u,$$

for some element $u \in H^{8k-1}(\hat{B}; Z_2)$. Recall that $\beta = \rho_2 \delta$, and that $2\delta = 0$. Thus, if we define

$$Q_{2k} = \bar{Q}_{2k} + \delta u,$$

we obtain an element $\in H^{8k}(\hat{B}; Z)$ such that by (5.4),

$$\pi^* P_{2k} = 2Q_{2k} + Q_k^2 - \pi^* \Phi_{2k}\,,$$

and

$$\rho_2 Q_{2k} = \pi^*(W_{8k} + \Psi_{8k}).$$

This completes the inductive definition of the classes Q_k and hence proves

[4] Since $\pi^* W_2 = 0$, we have $\pi^* \Omega_{8k} = \pi^* \hat{\Omega}_{8k}$.

(1.3) through (1.6). To prove (1.7) we apply (4.2) to the space $\hat{B}$, where we take the classes $Q_1, Q_2, \cdots$ to be the u's. (4.2)(1) and (2) are then fulfilled, since $K(Z_2, 1)$ has only cohomology of order 2. In order to obtain (4.2)(3) we make a change of variable. For $j \geq 4$, set

$$\hat{W}_j = \begin{cases} \pi^* W_j, & \text{if } j \text{ is not a power of 2.} \\ \pi^*(W_j + \Psi_j), & \text{if } j \text{ is a power of 2.} \end{cases}$$

Thus,

$$\beta \hat{W}_{2j} = \hat{W}_{2j+1}, \qquad \rho_2 Q_j = (\hat{W}_{2j})^2 \quad \text{if } j \text{ is not a power of 2,}$$

$$\beta \hat{W}_{2j} = 0, \qquad \rho_2 Q_l = \hat{W}_{2j} \quad \text{if } j = 2l, \text{ where } l \text{ is a power of 2.}$$

Since Ψ_j is decomposable, $H^*(\hat{B}; Z_2) = Z_2[\hat{W}_4, \hat{W}_6, \cdots]$, and therefore (4.3)(3) and (4) are fulfilled. Thus, (1.7) is simply (4.2) stated in terms of the Q's.

To complete the proof of (1.2), suppose that $\{Q_i'\}$ is a second set of cohomology classes satisfying properties (1.3)–(1.6), relative to a fixed choice of the classes Φ and Ψ. We show that $Q_i = Q_i'$, for $i \geq 1$. This is trivial if i is not a power of 2, and one readily verifies that it is so for $i = 1, 2$. Suppose we have shown that $Q_j = Q_j'$ for all integers j which are powers of $2 \leq 2^r (r \geq 1)$. As above, set $l = 2^{r-1}$, $k = 2l = 2^r$. We show that $Q_{2k} = Q_{2k}'$; which will complete the inductive argument.

By (1.5), $2(Q_{2k} - Q_{2k}') = 0$, and therefore by exactness,

$$Q_{2k} = Q_{2k}' + \delta u,$$

where $u \in H^{8k-1}(\hat{B}; Z_2)$. Now $2\,\delta u = 0$ and therefore $\delta u \in \hat{T}$ (see (1.7)). On the other hand, by (1.6),

$$0 = \rho_2(Q_{2k} - Q_{2k}') = \rho_2(\delta u).$$

But $\rho_2 \mid \hat{T}$ is a monomorphism, and therefore $\delta u = 0$, completing the proof of (1.2).

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