THE CHARACTERIZATION OF MOORE-POSTNIKOV INVARIANTS

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1. Introduction

In his application of Moore-Postnikov systems to the obstruction theory of fibre spaces, Hermann ([1], [2]) states and uses a theorem which gives two conditions characterizing the first non-vanishing k-invariant of a Moore-Postnikov system under certain assumptions. This theorem (4.1 of [1]), as stated there, is not quite correct. The purpose of this note is to present a correct version of Hermann's theorem and to give a counter-example to the original statement. Actually the theorem we state below is also a generalization of Hermann's theorem for it applies to any k-invariant, not merely the first non-vanishing one.

2. Characterizing Moore-Postnikov invariants

We follow Hermann's notation throughout this note. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibre space with simply-connected base and fibre, $0 < n(1) < n(2) < \cdots$ the dimensions in which F has non-zero homotopy groups, $A_j = \pi_{n(j)}(F)$. A Moore-Postnikov system of this fibre-space is a sequence of fibre maps $f_i : E_i \to E_{i-1}$ with fibre Eilenberg-MacLane spaces of type $(A_i, n(i))$, maps $h_i : E \to E_i$ such that $E_0 = B$, $h_0 = p$, $h_{i-1} = f_i h_i$, and h_i induces isomorphisms of homotopy groups in dimensions $\leqq n(i)$. Let g_i be $h_i \mid F$. We may assume that $h_{\alpha-1} : E \to E_{\alpha-1} (\alpha \geqq 1)$ is a fibre map with fibre F' and we have the commutative diagram

$$\begin{array}{ccccc}
F' & \xrightarrow{j_{\alpha-1}} & E & \xrightarrow{h_{\alpha-1}} & E_{\alpha-1} \\
\| & & \uparrow i & & \uparrow i_{\alpha-1} \\
F' & \xrightarrow{l_{\alpha-1}} & F & \xrightarrow{g_{\alpha-1}} & F_{\alpha-1}
\end{array}$$

where $F_{\alpha-1} = \text{image } (h_{\alpha-1} i)$, and $i_{\alpha-1}$, $j_{\alpha-1}$, $l_{\alpha-1}$ are inclusions. It is easily verified that the $F_{\alpha-1}$ are part of a Postnikov system for F.

The fibre F' is $n(\alpha) - 1$-connected and the spaces $E_{\alpha-1}$, $F_{\alpha-1}$ are 1-connected. Thus Serre's exact sequence and its naturality yield the following exact ladder (the coefficient group A_α will be systematically omitted from the notation):

$$\begin{array}{ccccccc}
H^{n(\alpha)+1}(E) & \xleftarrow{h^*_{\alpha-1}} & H^{n(\alpha)+1}(E_{\alpha-1}) & \xleftarrow{\tau_h} & H^{n(\alpha)}(F') & \xleftarrow{j^*_{\alpha-1}} & H^{n(\alpha)}(E) \\
\downarrow i^* & & \downarrow i^*_{\alpha-1} & & \| & & \downarrow i^* \\
H^{n(\alpha)+1}(F) & \xleftarrow{g^*_{\alpha-1}} & H^{n(\alpha)+1}(F_{\alpha-1}) & \xleftarrow{\tau_g} & H^{n(\alpha)}(F') & \xleftarrow{l^*_{\alpha-1}} & H^{n(\alpha)}(F)
\end{array}$$

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If b^α denotes the basic cohomology class of $H^{n(\alpha)}(F')$, and τ_h, τ_g the appropriate transgressions, then the αth k-invariants of p and of F are defined, respectively, by

$$k^\alpha = -\tau_h(b^\alpha)$$

$$k^\alpha(F) = -\tau_g(b^\alpha).$$

From these definitions and the above diagram, it immediately follows that

(1) $h_{\alpha-1}{}^*(k^\alpha) = 0$

(2) $i_{\alpha-1}{}^*(k^\alpha) = k^\alpha(F)$

We wish to study under which conditions, equations (1) and (2) characterize the k-invariant k^α.

LEMMA (2.1). *Let* $k \in H^{n(\alpha)+1}(E_{\alpha-1})$. *Then* k *satisfies* (1) *and* (2) *if and only if there is an element* $b \in H^{n(\alpha)}(F')$ *such that* $k = -\tau_h(b)$ *and* $b - b^\alpha \in$ image $l_{\alpha-1}{}^*$.

Proof. Suppose k satisfies (1) and (2). Then $h_{\alpha-1}{}^*(k) = 0$, so by exactness, there is an element $b \in H^{n(\alpha)}(F')$ such that $k = -\tau_h(b)$. Now, $-\tau_g(b^\alpha) = k^\alpha(F) = i_{\alpha-1}{}^*(k) = -i_{\alpha-1}{}^*\tau_h(b) = -\tau_g(b)$, so $\tau_g(b - b^\alpha) = 0$ and $b - b_\alpha \in$ image $l_{\alpha-1}{}^*$.

Conversely, if $k = -\tau_h(b)$ and $b - b^\alpha \in$ image $l_{\alpha-1}{}^*$, then $h_{\alpha-1}{}^*(k) = 0$ and $i_{\alpha-1}{}^*(k) = -i_{\alpha-1}{}^*\tau_h(b) = -\tau_g(b) = -\tau_g(b^\alpha) = k^\alpha(F)$.

LEMMA (2.2). *Let* $k \in H^{n(\alpha)+1}(E_{\alpha-1})$. *Then* $k = k^\alpha$ *if and only if there is an element* $b \in H^{n(\alpha)}(F')$ *such that* $k = -\tau_h(b)$ *and* $b - b^\alpha \in$ image $j_{\alpha-1}{}^*$.

Proof. If $k = -\tau_h(b)$, then $k = k^\alpha$ if and only if $\tau_h(b) = \tau_h(b^\alpha)$, i.e., $b - b^\alpha \in$ kernel $\tau_h =$ image $j_{\alpha-1}{}^*$, and the lemma is proved.

Combining lemmas (2.1) and (2.2) we obtain the following proposition.

PROPOSITION (2.3). *Let* $k \in H^{n(\alpha)+1}(E_{\alpha-1})$. *Then* $h_{\alpha-1}{}^*(k) = 0$, $i_{\alpha-1}{}^*(k) = k^\alpha(F)$ *imply* $k = k^\alpha$ *if and only if* image $l_{\alpha-1}{}^* \subset$ image $j_{\alpha-1}{}^*$.

COROLLARY (2.4). *Let* $k \in H^{n(\alpha)+1}(E_{\alpha-1})$. *Then* $h_{\alpha-1}{}^*(k) = 0$, $i_{\alpha-1}{}^*(k) = k^\alpha(F)$ *imply* $k = k^\alpha$ *if and only if* kernel $i_{\alpha-1}{}^* \cap$ image $\tau_h = (0)$.

Proof. Since $j_{\alpha-1}{}^* = l_{\alpha-1}{}^*i^*$, image $j_{\alpha-1}{}^* \subset$ image $l_{\alpha-1}{}^*$. Thus the second statement in proposition (2.3) is equivalent to image $l_{\alpha-1}{}^* =$ image $j_{\alpha-1}{}^*$. By exactness, this is equivalent to kernel $\tau_h =$ kernel τ_g. But kernel $\tau_g =$ kernel $i_{\alpha-1}{}^*\tau_h$ and the corollary follows.

In order to state our theorem concisely, we need the following definition.

DEFINITION (2.5). Let X be a space; G, an Abelian group; and $x \in H^*(X; G)$. Then $An(x)$, the *annihilator* of x is the subset of Hom (G, G) consisting of all homorphisms $\theta : G \to G$ such that $\theta_\#(x) = 0$, where $\theta_\#$ is the coefficient homomorphism induced by θ.

THEOREM (2.6). *Conditions* (1) *and* (2) *characterize* k^α *if and only if* $An(k^\alpha) = An(k^\alpha(F))$.

Proof. Let $x \in$ kernel $i_{\alpha-1}{}^* \cap$ image τ_h. Then $x = -\tau_h(b)$. Since $H^{n(\alpha)}(F'; A_\alpha) \approx \text{Hom}\,(H_{n(\alpha)}(F'); A_\alpha) \approx \text{Hom}\,(A_\alpha, A_\alpha)$, it is readily seen that b is of the form $\theta_{\#}(b^\alpha)$ for some $\theta \in \text{Hom}\,(A_\alpha, A_\alpha)$—in fact θ is precisely the homomorphism corresponding to b under the above isomorphisms. Then $x = -\tau_h\theta_{\#}(b^\alpha) = -\theta_{\#}\tau_h(b^\alpha) = \theta_{\#}k^\alpha$ and $0 = i_{\alpha-1}{}^*(x) = i_{\alpha-1}{}^*(\theta_{\#}k^\alpha) = \theta_{\#}i_{\alpha-1}{}^*(k^\alpha) = \theta_{\#}k^\alpha(F)$. The theorem now follows from (2.4).

The above theorem may be used to compute k^α from the knowledge of $k^\alpha(F)$, provided properties of k^α are eliminated from the statement "$An(k^\alpha) = An(k^\alpha(F))$." Since $An(k^\alpha) \subset An(k^\alpha(F))$, the equality automatically holds in the special case where $An(k^\alpha(F)) = (0)$. This is true, as is easily verified under the conditions stated in the following two corollaries.

COROLLARY (2.7). *If* A_α *is the field of rational numbers* Q *or* Z_p (p prime) *and* $k^\alpha(F) \neq 0$, *then* (1) *and* (2) *characterize* k^α.

COROLLARY (2.8). *If* A_α *and* $H^{n(\alpha)+1}(F_{\alpha-1}; A_\alpha)$ *are free and* $k^\alpha(F) \neq 0$, *then* (1) *and* (2) *characterize* k^α.

These corollaries cover the cases where Hermann applied his theorem.

3. An example

The example described below shows that $An(k^\alpha)$ may be different from $An(k^\alpha(F))$ if $A_\alpha \approx Z$ and no further restrictions are made. Thus the statement of Theorem 4.1 of [1] must be modified.

Let $B = S^7$, the 7-sphere, and $\pi_i(F) \neq 0$ only for $i = 4, 6$, when $\pi_4(F) \approx Z_2$, $\pi_6(F) \approx Z$. Suppose the first k-invariant vanishes so that $E_1 = S^7 \times K(Z_2, 4)$, $k^1(F)$ is the non-zero element of $H^7(Z_2, 4; Z) \approx Z_2$, $\delta^* \text{Sq}^2 \cdot b^4$, and $k^1 \in H^7(E_1; Z) \approx H^7(S^7; Z) + H^7(Z_2, 4; Z) \approx Z + Z_2$ is the element $s^7 + \delta^* \text{Sq}^2 b^4$, where s^7 is a generator of $H^7(S^7; Z)$. It is easily seen that all elements $ns^7 + \delta^* \text{Sq}^2 b^4$, n odd, satisfy conditions (1) and (2). This reflects the fact that $An(k^1) = (0)$ while $An(k^1(F)) = 2Z$.

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