ON THE DUAL STIEFEL-WHITNEY CLASSES OF A MANIFOLD

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1. Introduction

This paper is devoted to proving theorems which assert that for certain values of n and k, the $(n - k)$-dimensional dual Stiefel-Whitney class of an n-dimensional manifold is always zero. The precise statements of the theorems are given in §2, and their proofs are given in §§3–8.

This work was originally motivated by the fact that one of the most potent methods of proving that a manifold can not be imbedded in certain dimensional Euclidean spaces is to prove that certain dual Stiefel-Whitney classes of the manifold are non-zero. Thus it is natural to investigate the limitations of this method of proving non-imbeddability.

Later on, this work was given additional stimulus by the remarkable results of A. Haefliger and M. Hirsch on the imbeddability of differentiable manifolds in Euclidean space. Haefliger has proved in [7] that a compact n-dimensional $(k - 1)$-connected manifold can always be imbedded differentiably in Euclidean space of dimension $2n + 1 - k$, provided $2k < n$. In case $2(k + 1) < n$ Haefliger and Hirsch have shown[2] that the first "obstruction" to imbedding the manifold in $(2n - k)$-dimensional Euclidean space is the dual Stiefel-Whitney class in dimension $n - k$ (taken with mod 2 coefficients for $(n - k)$ even, and integral or twisted integral coefficients for $(n - k)$ odd). The vanishing of this obstruction is both necessary and sufficient for the desired imbedding. Thus our results show that for certain values of n and k, this desired imbedding is always possible. Among the more striking results obtained by combining our theorems with those of Haefliger and Hirsch are the following:

(1) *A compact, orientable differentiable n-manifold can be imbedded differentiably in Euclidean* $(2n - 1)$*-space* (with the possible exception of the case $n = 4$).

(2) *If n is not a power of* 2, *then any compact, non-orientable n-manifold can be differentiably imbedded in Euclidean* $(2n - 1)$*-space* (with the possible exception of the case $n = 3$).

(3) *A compact, simply connected n-manifold can be differentiably imbedded in Euclidean* $(2n - 2)$*-space provided n is not of the form* 2^k *or* $2^k + 1$ (with the possible exception of the case $n = 6$).

Independently of the imbedding problem, it seems worthwhile to study the Stiefel-Whitney classes of a differentiable manifold, since they are among the most important algebraic invariants of such a manifold. The results in this

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[2] This result is unpublished as yet.

paper are a small contribution toward the difficult problem of determining all the relations which must be satisfied by the Stiefel-Whitney classes of a manifold. (Added in proof: This problem has recently been solved by E. H. Brown, Jr. and F. P. Peterson.)

The results in this paper include the results on dual Stiefel-Whitney classes contained in two earlier notes by one of the authors ([1] and [2]). This paper is written so that a knowledge of these earlier papers is not necessary. It should be pointed out, however, that the results on *ordinary* Stiefel-Whitney classes of a manifold given in [1] and [2] are *not* contained in the present paper.

2. Statement of Results

First we list the notations which are fixed throughout the paper.

M^n—compact, connected, differentiable, n-dimensional manifold.

w_i, $\bar{w}_i$—mod 2 i-dimensional Stiefel-Whitney class and dual Stiefel-Whitney class of M^n respectively.

W_i, $\bar{W}_i$—integral i-dimensional Stiefel-Whitney class and dual Stiefel-Whitney class of M^n respectively. These are defined for i odd and $i > 2$ only; if M^n is non-orientable, they have *twisted* integer coefficients, while if M^n is orientable, they are cohomology classes with ordinary integer coefficients. They may be defined by the following equations:

$$W_{2i+1} = \delta(w_{2i}),$$
$$\bar{W}_{2i+1} = \delta(\bar{w}_{2i}),$$

where δ denotes the appropriate Bockstein coboundary operator (see §7 for more details).

Let $\alpha(n)$ denote the number of occurrences of the digit "one" when the integer n is written in its dyadic expansion.

THEOREM I. *For any* M^n,
(i) $\bar{w}_{n-k} = 0$ *for* $0 \leqq k < \alpha(n)$, *and*
(ii) $\bar{W}_{n-k} = 0$ *for* $0 \leqq k < \alpha(n)$ *and* $n - k$ *odd.*

This theorem is "best possible" in the following sense: For any integer $n > 1$ there exists a manifold M^n such that $\bar{w}_{n-k} \neq 0$ for $\alpha(n) \leqq k \leqq n$. The manifold M^n is a product of real projective spaces chosen as follows. Write the integer n as a sum of $\alpha(n)$ distinct powers of 2:

$$n = 2^{a_1} + 2^{a_2} + \cdots, \quad a_1 < a_2 < \cdots .$$

Let X_i denote a real projective space of dimension 2^{a_i}, $1 \leqq i \leqq \alpha(n)$. Then set

$$M^n = X_1 \times \cdots \times X_{\alpha(n)} .$$

It is readily proved that $\bar{w}_{n-k} \neq 0$ for $\alpha(n) \leqq k \leqq n$ for this choice of M^n. Since $\bar{w}_{n-k}$ is the reduction mod 2 of $\bar{W}_{n-k}$, it follows that $\bar{W}_{n-k} \neq 0$ for $\alpha(n) \leqq k < n$ and $n - k$ odd. This example was pointed out to the authors by Rolph Schwarzenberger.

In the above example, M^n is non-orientable. For orientable manifolds, further results can be obtained.

THEOREM II. *Let M^n be a compact, connected, orientable n-dimensional manifold such that $\bar{w}_{n-k} \neq 0$. Then there exist non-negative integers $\alpha_0, \alpha_1, \cdots, \alpha_r$ which satisfy the following conditions*:

(a) $\sum_j \alpha_j = k$,

(b) $\sum_j 2^j \alpha_j = n$,

(c) α_1 *is even, and*

(d) *If $\alpha_0 = 0$, then the first non-zero α_j and its immediate successor, α_{j+1}, must be even.*

COROLLARY 1. *For any orientable n-manifold, $\bar{w}_{n-1} = 0$.*
This is the case $k = 1$ of the theorem.

COROLLARY 2. *If M^n is an orientable n-manifold $n > 2$, such that $\bar{w}_{n-2} \neq 0$, then n has the form $n = 2^r + 1$, $r > 1$, or $n = 2^{r+1}$, $r > 0$.*

Proof: Take $k = 2$ in Theorem II. The only way to satisfy conditions (a)–(d) for $n > 2$ is to choose $\alpha_0 = \alpha_r = 1$, $\alpha_j = 0$ for $0 < j < r$, $r > 1$, or to choose $\alpha_r = 2$, $\alpha_j = 0$ for $j < r$, $r > 0$. In the first case, $n = 2^r + 1$, $r > 1$, and in the second case $n = 2^{r+1}$, $r > 0$. Q.E.D.

It should be noted that both these cases can actually occur: Complex projective space of real dimension $n = 2^{r+1}$ illustrates the second case, and the manifold which Dold ([5]) denotes by $P(1, 2^{r-1})$, $r > 1$, illustrates the first case.

The arguments in the foregoing paragraphs can be construed as proving that Theorem II is "best possible" for $k = 1$ or 2. On the other hand, it is not known at present whether or not this theorem is a best possible result for $k = 3$. However, the following examples[3] show that Theorem II is "best possible" for $k = 3$ when $n \leqq 11$: $M^5 = P_2(\mathbf{C}) \times S^1$, $M^6 = P(1, 2) \times S^1$, $M^9 = P_4(\mathbf{C}) \times S^1$, and $M^{10} = P(1, 4) \times S^1$. The cases $n = 7$ or 11 are covered by Theorem II (c) and $n = 8$ is covered by Theorem II (d). The first unsolved case when $k = 3$ is $n = 12$.

THEOREM III. *Let M^n be a compact, connected, orientable manifold such that $\bar{W}_{n-k} \neq 0$, $n - k$ odd. Then there exist non-negative integers $\alpha_0, \alpha_1, \cdots, \alpha_r$ such that*

(a) $\sum_j \alpha_j = k + 1$,

(b) $\sum_j 2^j \alpha_j = n$,

(c) α_1 *is even,*

(d) If $\alpha_0 = 0$, *then the first non-zero α_j and its immediate successor must be even,*

(e) $1 < \alpha_r < k + 1$.

[3] In these examples, $P_n(\mathbf{C})$ denotes complex projective space of $2n$ real dimensions, and $P(m, n)$ is the Dold manifold ([5]).

Note that conditions (b), (c), and (d) are the same in both Theorems II and III.

The cases $k = 1$ and $k = 2$ are of special interest:

COROLLARY 1. *If M^n is an orientable n-manifold, n even, then* $\bar{W}_{n-1} = 0$.

COROLLARY 2. *If M^n is an orientable n-manifold, n odd, such that* $\bar{W}_{n-2} \neq 0$, *then n is of the form* $n = 2^{r+1} + 1$, $r > 0$.

An example of such a manifold is the Dold manifold $P(1, 2^r)$.

3. Relationship between Steenrod Squares and Dual Stiefel-Whitney Classes

Notation: $\mathrm{Sq} = \sum_{i \geqq 0} \mathrm{Sq}^i$ denotes the "total" Steenrod square.

It is known that Sq is an *automorphism* of the mod 2 cohomology ring $H^*(X, \mathbf{Z}_2)$ of any space X such that $H^q(X, \mathbf{Z}_2) = 0$ for all sufficiently large q (for a proof, see §I of Dold, [6]). Following Thom, we will denote the inverse automorphism by

$$Q = 1 + Q^1 + Q^2 + Q^3 + \cdots,$$

where Q^i denotes the component of degree i of Q. The relation

$$Q \circ \mathrm{Sq} = \mathrm{Sq} \circ Q = 1$$

implies that for any positive integer k,

$$\sum_{i+j=k} Q^i \mathrm{Sq}^j = 0.$$

The cohomology operations Q^i may be readily computed from these equations; for example,

$$Q^1 = \mathrm{Sq}^1, \qquad Q^2 = \mathrm{Sq}^2,$$
$$Q^3 = \mathrm{Sq}^2\mathrm{Sq}^1, \qquad Q^4 = \mathrm{Sq}^4 + \mathrm{Sq}^3\mathrm{Sq}^1,$$
$$Q^5 = \mathrm{Sq}^4\mathrm{Sq}^1, \qquad Q^6 = \mathrm{Sq}^4\mathrm{Sq}^2.$$

LEMMA 1. *Let M^n be a compact, connected n-manifold. Then for any* $x \in H^k(M^n, \mathbf{Z}_2)$,

$$x\bar{w}_{n-k} = Q^{n-k}(x).$$

Proof: In the proof of this lemma, we will use the following notation:

$w = \sum_{i \geqq 0} w_i$ (total Stiefel-Whitney class of M^n).

$\bar{w} = \sum_{i \geqq 0} \bar{w}_i$ (total dual Stiefel-Whitney class). Note that $\bar{w} = w^{-1}$.

$\mu \in H_n(M^n, \mathbf{Z}_2)$ (fundamental homology class of M^n). If y is any element of the cohomology ring $H^*(M^n, \mathbf{Z}_2)$ (it need not be a homogeneous element), then we will denote by $\langle y, \mu \rangle$ the element of $\mathbf{Z}_2$ obtained by evaluating the component of y of degree n on the homology class μ. With this convention, the Wu class ([11])

$$U = 1 + U_1 + U_2 + \cdots$$

of the manifold M^n is defined by the requirement that the relation

$$\langle x \cdot U, \mu \rangle = \langle \mathrm{Sq}\, x, \mu \rangle$$

shall be true for *any* $x \in H^*(M^n, \mathbf{Z}_2)$. It is known (see [11]) that

$$w = \mathrm{Sq}\, U.$$

Hence

$$\bar{w} = w^{-1} = \mathrm{Sq}(U^{-1})$$

and for any $x \in H^*(M^n, \mathbf{Z}_2)$,

$$\begin{aligned} x \cdot \bar{w} &= x \cdot w^{-1} = x \cdot \mathrm{Sq} U^{-1} \\ &= \mathrm{Sq}(Qx) \cdot \mathrm{Sq} U^{-1} \\ &= \mathrm{Sq}[(Qx) \cdot U^{-1}]. \end{aligned}$$

Therefore

$$\begin{aligned} \langle x \cdot \bar{w}, \mu \rangle &= \langle \mathrm{Sq}[(Qx) \cdot U^{-1}], \mu \rangle \\ &= \langle U \cdot [(Qx) \cdot U^{-1}], \mu \rangle \\ &= \langle Qx, \mu \rangle, \end{aligned}$$

and from the fact that $\langle x \cdot \bar{w}, \mu \rangle = \langle Qx, \mu \rangle$, the statement of the lemma follows immediately. Q.E.D.

Remarks: (1) The authors are indebted to John Milnor for this simplified proof of this lemma. (2) Milnor has shown that the Steenrod algebra (mod 2) admits a canonical involution, and that the image of Sq under this involution is precisely the operator Q. This fact will not figure in this paper, however.

4. Iterated Steenrod Squares

If $I = (i_1, i_2, \cdots, i_r)$ is any sequence of non-negative integers, then Sq^I denotes the iterated Steenrod square $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r}$. The sequence I is said to be *admissible* if $i_1 \geqq 2i_2, i_2 \geqq 2i_3 \cdots, i_{r-1} \geqq 2i_r$. By use of the Adem relations, any iterated Steenrod square may be expressed as a sum of admissible iterated Steenrod squares. Following Serre ([9]), it is convenient to associate with any admissible sequence $I = (i_1, i_2, \cdots, i_r)$ the sequence $(\alpha_1, \cdots, \alpha_r)$ of non-negative integers defined by

$$\begin{aligned} \alpha_1 &= i_1 - 2i_2, \\ \alpha_2 &= i_2 - 2i_3 \\ &\vdots \\ \alpha_{r-1} &= i_{r-1} - 2i_r \\ \alpha_r &= i_r. \end{aligned} \tag{1}$$

Clearly, the sequence $(\alpha_1, \cdots, \alpha_r)$ uniquely determines the sequence $(i_1, \cdots, i_r)$. The *degree* of the sequence I, $n(I)$, is

$$n(I) = i_1 + i_2 + \cdots + i_r,$$

and the *excess* of the admissible sequence I, $e(I)$, is

$$e(I) = \alpha_1 + \alpha_2 + \cdots + \alpha_r.$$

The following relationship involving $n(I)$ and $e(I)$ may be easily verified:

$$n(I) + e(I) = 2i_1. \tag{2}$$

LEMMA 2. *If* Sq^I *is an admissible iterated Steenrod square and* x *is any mod 2 cohomology class such that degree* $x < e(I)$, *then* $\mathrm{Sq}^I x = 0$.

This follows easily from the fact that $\mathrm{Sq}^k y = 0$ if degree $y < k$.

Next, we will consider the cohomology class $\mathrm{Sq}^I x$, where I is an admissible sequence, and x is a mod 2 cohomology class of degree q. We may as well assume that $e(I) \leqq q$ in view of Lemma 2. In this case, it is convenient to define

$$\alpha_0 = q - e(I); \quad \text{i.e., } \alpha_0 + \alpha_1 + \cdots + \alpha_r = q. \tag{3}$$

Then

$$\begin{aligned} \text{degree } (\mathrm{Sq}^I x) &= n(I) + q \\ &= 2i_1 - e(I) + q \quad \text{(by equation (2))} \\ &= 2i_1 + \alpha_0 \\ &= 2^r\alpha_r + 2^{r-1}\alpha_{r-1} + \cdots + 2\alpha_1 + \alpha_0 \end{aligned} \tag{4}$$

since $i_1 = 2^{r-1}\alpha_r + 2^{r-2}\alpha_{r-1} + \cdots + \alpha_1$, as may be seen from equations (1).

LEMMA 3. *Using the above notation, assume that* $\alpha_0 = \alpha_1 = \cdots = \alpha_{j-1} = 0$ $(j > 0)$. *Then*

$$\mathrm{Sq}^I x = [\mathrm{Sq}^J x]^{2^j},$$

where

$$J = (i_{j+1}, i_{j+2}, \cdots, i_r).$$

The proof is made by repeated application of the fact that $\mathrm{Sq}^k y = y^2$ if degree $y = k$.

5. Proof of Part of Theorem I

LEMMA 4. *Let* M^n *be a compact, connected n-manifold. If* $\bar{w}_{n-k} \neq 0$, *then there exist non-negative integers* $\alpha_0, \alpha_1, \cdots, \alpha_r$ *such that* $\sum_j \alpha_j = k$ *and* $\sum_j 2^j\alpha_j = n$.

Proof: By the Poincaré Duality Theorem (mod 2), there exists an $x \in H^k(M^n, \mathbf{Z}_2)$ such that

$$x \cdot \bar{w}_{n-k} \neq 0.$$

By Lemma 1,

$$Q^{n-k}(x) \neq 0.$$

Now Q^{n-k} may be expressed as a sum of admissible iterated Steenrod squares; hence there exists an admissible iterated Steenrod square, Sq^I, of degree $n - k$ such that

$$\mathrm{Sq}^I(x) \neq 0.$$

By Lemma 2 we know that $e(I) \leqq k$. If, now, we write down the analogue of equation (4) for the degree of $\mathrm{Sq}^I x$ we obtain precisely the conclusions of this lemma. Q.E.D.

The part of Theorem I concerned with the mod 2 dual Stiefel-Whitney classes is essentially the contrapositive form of the statement of this lemma. To see this, observe that for any positive integers n and k, there exist non-negative integers $\alpha_0, \alpha_1, \cdots, \alpha_r$ such that $\sum \alpha_j = k$ and $\sum 2^j \alpha_j = n$ if and only if $n \geqq k \geqq \alpha(n)$.

6. Proof of Theorem II

Recall that a compact, connected n-manifold M^n is orientable if and only if the homomorphism $\mathrm{Sq}^1 : H^{n-1}(M^n, \mathbf{Z}_2) \to H^n(M^n, \mathbf{Z}_2)$ is zero (this follows from the known structure of $H^n(M^n, \mathbf{Z})$ and the fact that $\mathrm{Sq}^1 = \rho \circ \delta$ in the notation of §7). As a consequence, *if M^n is orientable and i is an odd integer, the homomorphism $\mathrm{Sq}^i : H^{n-i}(M^n, \mathbf{Z}_2) \to H^n(M^n, \mathbf{Z}_2)$ is zero*; for, $\mathrm{Sq}^i = \mathrm{Sq}^1\mathrm{Sq}^{i-1}$ if i is odd. This is the only property of orientable manifolds which we use to prove Theorems II and III.

One starts the proof of Theorem II like that of Lemma 4. If M^n is compact, connected, orientable, and $\bar{w}_{n-k} \neq 0$, then, exactly as in the proof of Lemma 4, one concludes that there exists an $x \in H^k(M^n, \mathbf{Z}_2)$ and an admissible sequence $I = (i_1, i_2, \cdots, i_r)$ of degree $n - k$ such that $\mathrm{Sq}^I x \neq 0$. Conditions (a) and (b) of Theorem II are exactly the conclusion of Lemma 4, and the proof is the same. However, since M^n is orientable, we can conclude that i_1 is even by the remark in the preceding paragraph. Since $\alpha_1 = i_1 - 2i_2$, this implies that α_1 is even, which is part (c) of the conclusion.

Next we will prove part (d) of the conclusion. If $\alpha_0 = \alpha_1 = \cdots = \alpha_{j-1} = 0$, then by Lemma 3,

$$\begin{aligned}\mathrm{Sq}^I(x) &= [\mathrm{Sq}^J x]^{2^j} \\ &= [\mathrm{Sq}^K x]^{2^{j-1}},\end{aligned}$$

where $J = (i_{j+1}, i_{j+2}, \cdots, i_r)$ and $K = (i_j, i_{j+1}, \cdots, i_r)$. Note also that it follows from equations (1) that $\alpha_{j+1} \equiv i_{j+1}$ and $\alpha_j \equiv i_j \bmod 2$. If α_j is odd, then so is i_j, and

$$\mathrm{Sq}^I x = \mathrm{Sq}^1\{\mathrm{Sq}^{K'} x [\mathrm{Sq}^K x]^{2^{j-1}-1}\}$$

where $K' = (i_j - 1, i_{j+1}, \cdots, i_r)$, since Sq^1 is a derivation of $H^*(M^n, \mathbf{Z}_2)$. If α_{j+1} is odd, then so is i_{j+1}, and

$$\mathrm{Sq}^I x = \mathrm{Sq}^1\{\mathrm{Sq}^{J'} x [\mathrm{Sq}^J x]^{2^j-1}\}.$$

where $J' = (i_{j+1} - 1, i_{j+2}, \cdots, i_r)$. In either case, this contradicts the fact that $\mathrm{Sq}^I x \neq 0$.

This completes the proof of Theorem II.

7. Some Consequences of the Poincaré Duality Theorem

For any manifold M we will denote by $T^q(M, \mathbf{Z})$ and $T_q(M, \mathbf{Z})$ the torsion subgroups of $H^q(M, \mathbf{Z})$ and $H_q(M, \mathbf{Z})$ respectively. Also, for any prime number p, let

$$(S) \qquad H^q(M, \mathbf{Z}) \xrightarrow{p} H^q(M, \mathbf{Z}) \xrightarrow{\rho} H^q(M, \mathbf{Z}_p) \xrightarrow{\delta} H^{q+1}(M, \mathbf{Z})$$

denote the exact sequence associated with the coefficient sequence

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}_p \to 0.$$

First, we will consider the case of a compact, connected n-manifold M^n which is *orientable.* The cup product

$$H^q(M^n, \mathbf{Z}_p) \times H^{n-q}(M^n, \mathbf{Z}_p) \to H^n(M^n, \mathbf{Z}_p) \approx \mathbf{Z}_p$$

is a bilinear form, which, according to the Poincaré Duality Theorem, is non-degenerate.

LEMMA 5. *For any integer q, $\rho[T^{n-q}(M^n, \mathbf{Z})]$ is the annihilator of $\rho[H^q(M^n, \mathbf{Z})]$.*

Apparently this lemma is well known; see, for example, Hopf and Hirzebruch [8], page 169. A proof can be easily given along the lines of the proof of Lemma 7 below.

Next, we will consider the analogous statements for a compact, connected *non-orientable* manifold M^n. For this purpose, it is necessary to consider homology and cohomology groups with twisted integer coefficients (denoted by $\mathcal{Z}$) or twisted integer mod p coefficients (denoted by $\mathcal{Z}_p$). These are local coefficient systems in M^n such that an orientation reversing element of the fundamental group operates on $\mathbf{Z}$ on $\mathbf{Z}_p$ by change of sign. Note that the homology and cohomology groups with coefficients $\mathbf{Z}_p$ or $\mathcal{Z}_p$ are vector spaces over the field $\mathbf{Z}_p$, and that for $p = 2$, the local coefficient system $\mathcal{Z}_2$ is trivial, i.e. $\mathcal{Z}_2 = \mathbf{Z}_2$ as coefficient system. We assume the reader is familiar with the Poincaré Duality Theorem for compact, non-orientable manifolds using twisted coefficients as outlined, for example, in [3], exposé 20; [4], exposé XVII; or Steenrod ([10]), §§14–15. Recall that the cup product of two twisted cohomology classes is an ordinary (non-twisted) cohomology class, while the product of a twisted and an ordinary cohomology class is a cohomology class with twisted coefficients. A similar remark applies to the cap product of a homology and a cohomology class, one or both of which may have twisted coefficients.

The following lemma is probably well-known, but to the best of the authors' knowledge does not occur in the literature.

LEMMA 6. *The cup product*

$$\smile : H^q(M^n, \mathbf{Z}_p) \times H^{n-q}(M^n, \mathcal{Z}_p) \rightarrow H^n(M^n, \mathcal{Z}_p) \approx \mathbf{Z}_p$$

is a non-degenerate bilinear form.

Proof: Let $\mu \in H_n(M^n, \mathcal{Z})$ denote the fundamental homology class of M^n. According to the Poincaré Duality Theorem for non-orientable manifolds, the homomorphisms

$$H^q(M^n, \mathcal{Z}_p) \rightarrow H_{n-q}(M^n, \mathbf{Z}_p),$$

$$H^q(M^n, \mathbf{Z}_p) \rightarrow H_{n-q}(M^n, \mathcal{Z}_p),$$

defined by $x \rightarrow x \frown \mu$ for $x \in H^q(M^n, \mathcal{Z}_p)$ or $H^q(M^n, \mathbf{Z}_p)$ are isomorphisms onto. Now consider the following diagram:

$$\begin{array}{ccc} H^q(M^n, \mathbf{Z}_p) \times H^{n-q}(M^n, \mathcal{Z}_p) & \xrightarrow{1} & H^n(M^n, \mathcal{Z}_p) \\ \downarrow 3 \qquad\qquad \downarrow 4 & & \downarrow 5 \\ H^q(M^n, \mathbf{Z}_p) \times H_q(M^n, \mathbf{Z}_p) & \xrightarrow{2} & H_0(M^n, \mathbf{Z}_p) \end{array}$$

Here arrows nos. 1 and 2 denote the cup product and cap product respectively, arrow no. 3 denotes the identity map, and arrows nos. 4 and 5 denote cup products with the fundamental class μ. The fact that this diagram is commutative follows from the well known "associative" law,

$$u \frown (v \frown \mu) = (u \smile v) \frown \mu.$$

Since the vertical arrows are all isomorphisms onto, and the bilinear form denoted by arrow no. 2 is well known to be non-degenerate for any connected finite simplicial complex, it follows that the bilinear form denoted by arrow no. 1 is also non-degenerate. Q.E.D.

For the statement of the next lemma, we will use the following notation: The torsion subgroups of $H^q(M^n, \mathcal{Z})$, and $H_q(M^n, \mathcal{Z})$ will be denoted by $T^q(M^n, \mathcal{Z})$, and $T_q(M^n, \mathcal{Z})$ respectively; the exact coefficient sequence

$$0 \rightarrow \mathcal{Z} \xrightarrow{p} \mathcal{Z} \rightarrow \mathcal{Z}_p \rightarrow 0,$$

gives rise to a corresponding exact sequence in the cohomology of M^n:

$$(S') \quad H^q(M^n, \mathcal{Z}) \xrightarrow{p} H^q(M^n, \mathcal{Z}) \xrightarrow{\rho'} H^q(M^n, \mathcal{Z}_p) \xrightarrow{\delta'} H^{q+1}(M^n, \mathcal{Z}).$$

Lemma 7. *With respect to the bilinear form of Lemma* 6, *the subspaces orthogonal to* $\rho[H^q(M^n, \mathbf{Z})] \subset H^q(M^n, \mathbf{Z}_p)$ *and* $\rho'[H^{n-q}(M^n, \mathcal{Z})] \subset H^{n-q}(M^n, \mathcal{Z}_p)$ *are* $\rho'[T^{n-q}(M^n, \mathcal{Z})]$ *and* $\rho[T^q(M^n, \mathbf{Z})]$, *respectively.*

Proof: Since ρ and ρ' commute with cup products (i.e., $(\rho u) \smile (\rho' v) = \rho'(u \smile v)$ for $u \in H^q(M^n, \mathbf{Z})$ and $v \in H^{n-q}(M^n, \mathcal{Z})$) and $H^n(M^n, \mathcal{Z})$ is infinite cyclic, it is clear that $\rho'[T^{n-q}(M^n, \mathcal{Z})]$ is contained in the subspace orthogonal to $\rho[H^q(M^n, \mathbf{Z})]$ and that $\rho[T^q(M^n, \mathbf{Z})]$ is contained in the subspace orthogonal to $\rho'[H^{n-q}(M^n, \mathcal{Z})]$. To complete the proof, one computes the dimension of these subspaces and shows that they have the correct dimensions for orthogonal subspaces. Let

$$b_i = \text{rank } H^i(M^n, \mathbf{Z}),$$

$$b_i' = \text{rank } H^i(M^n, \mathcal{Z}),$$

$$c_i = \text{number of cyclic summands in the } p\text{-primary component of } T^i(M^n, \mathbf{Z}),$$

$$c_i' = \text{number of cyclic summands in the } p\text{-primary component of } T^i(M^n, \mathcal{Z}).$$

Then $b_i = b_{n-i}'$, since $H^{n-i}(M^n, \mathcal{Z}) \approx H_i(M^n, \mathbf{Z})$ by the Poincaré Duality theorem, and $H_i(M^n, \mathbf{Z})$ has the same rank as $H^i(M^n, \mathbf{Z})$ for any finite simplicial complex. Similarly, $c_i = c_{n-i+1}'$, since $T^{n-i+1}(M^n, \mathcal{Z}) \approx T_{i-1}(M^n, \mathbf{Z})$ by Poincaré duality and $T_{i-1}(M^n, \mathbf{Z}) \approx T^i(M^n, \mathbf{Z})$ for any finite polyhedron.

Consideration of the exact sequence (S') shows that the rank of the vector space $H^q(M, \mathbf{Z}_p)$ over the field $\mathbf{Z}_p$ is $b_q + c_q + c_{q+1} = b_{n-q}' + c_{n-q+1}' + c_{n-q}'$ which is also the rank of $H^{n-q}(M, \mathcal{Z}_p)$ over $\mathbf{Z}_p$.
One also easily determines the following:

$$\text{rank } \{\rho H^q(M, \mathbf{Z})\} = b_q + c_q,$$

$$\text{rank } \{\rho' H^{n-q}(M, \mathcal{Z})\} = b_{n-q}' + c_{n-q}',$$

$$\text{rank } \{\rho T^q(M, \mathbf{Z})\} = c_q = c_{n-q+1}', \quad \text{and}$$

$$\text{rank } \{\rho' T^{n-q}(M, \mathcal{Z})\} = c_{n-q}' = c_{q+1}.$$

From these statements the lemma follows.

We will now apply these results in the case $p = 2$.

Lemma 8. *For any compact, connected n-manifold M^n, orientable or non-orientable,* $\bar{W}_{n-k} = 0$ *if and only if*

$$\bar{w}_{n-k-1} \cdot x = 0$$

for any $x \in \rho[T^{k+1}(M^n, \mathbf{Z})]$.

Proof: If M^n is orientable,

$$\bar{W}_{n-k} = \delta(\bar{w}_{n-k-1});$$

while if M^n is non-orientable,

$$\bar{W}_{n-k} = \delta'(\bar{w}_{n-k-1}).$$

Thus $\bar{W}_{n-k} = 0$ if and only if $\bar{w}_{n-k-1}$ belongs to

$$\rho[H^{n-k-1}(M^n, \mathbf{Z})] \quad \text{or} \quad \rho'[H^{n-k-1}(M^n, \mathcal{Z})],$$

respectively; by Lemmas 5 and 7, this is equivalent to $\bar{w}_{n-k-1}$ annihilating $\rho[T^{k+1}(M^n, \mathbf{Z})]$ in either the orientable or non-orientable case. Q.E.D.

COROLLARY. *For any compact, connected n-manifold M^n, orientable or non-orientable, $\bar{W}_{n-k} = 0$ if and only if $Q^{n-k-1}(x) = 0$ for any $x \in \rho[T^{k+1}(M^n, \mathbf{Z})]$.*

Proof: Use Lemma 1.

8. Proof of the Last Part of Theorem I and of Theorem III

First, one proves the following assertion:

LEMMA 9. *Let M^n be a compact connected manifold, orientable or non-orientable, such that $\bar{W}_{n-k} \neq 0$ ($n - k$ odd). Then there exist non-negative integers α_0, α_1, $\cdots$, α_r such that $\alpha_r > 1$, $\sum_j \alpha_j = k + 1$,* and *$\sum_j 2^j\alpha_j = n$.*

Proof: It follows from the corollary to Lemma 8 that there exists an element $x \in \rho[T^{k+1}(M^n, \mathbf{Z})]$ and an admissible sequence $I = (i_1, i_2, \cdots, i_r)$ of degree $n - k - 1$ such that

$$\mathrm{Sq}^I(x) \neq 0.$$

From this it follows that $i_r > 1$, since $\mathrm{Sq}^1 x = 0$ because x is the reduction mod 2 of an integral cohomology class. Also, $e(I) \leqq k + 1$ by Lemma 2. If now one applies the analog of equation (4) in this situation, the desired conclusion is obtained (note that $\alpha_r = i_r$). Q.E.D.

LEMMA 10. *Let n and k be positive integers such that $k < n$ and $n - k$ is odd. Then the following two conditions on the pair $[n, k]$ are equivalent:*

(a) There exist non-negative integers α_0, α_1, $\cdots$, α_r ($r > 0$) such that $\sum_j \alpha_j = k$ and $\sum 2^j\alpha_j = n$.

(b) There exist non-negative integers β_0, β_1, $\cdots$, β_s ($s > 0$) such that $\beta_s > 1$, $\sum_j \beta_j = k + 1$, and $\sum_j 2^j\beta_j = n$.

Proof: First we prove that (b) implies (a). Given β_0, β_1, $\cdots$, β_s satisfying condition (b), define $\alpha_j = \beta_j$ for $0 \leqq j \leqq s - 1$, $\alpha_s = \beta_s - 2$, $\alpha_{s+1} = 1$. Then the integers α_0, α_1, $\cdots$, α_{s+1} satisfy condition (a).

Next, we prove that (a) implies (b). Assume α_0, $\cdots$, α_r are given satisfying condition (a). Without loss of generality, we may assume $\alpha_r > 0$, for this can always be achieved by a suitable change of notation. There are now three cases to consider.

Case 1: $\alpha_r = 1$. Then choose $s = r - 1$, $\beta_j = \alpha_j$ for $0 \leqq j \leqq r - 2$, $\beta_{r-1} = \alpha_{r-2} + 2$, $\beta_r = 0$.

Case 2: $\alpha_r = 2$. It is easily seen that the hypothesis that $n - k$ is odd implies that there exists an index t such that $0 < t < r$ and $\alpha_t > 0$. Define $\beta_{t-1} = \alpha_{t-1} + 2$,

$\beta_t = \alpha_t - 1$, and $\beta_j = \alpha_j$ for all other values of j, $0 \leqq j \leqq r$. (This is the only place where the hypothesis that $n - k$ is odd is used).

Case 3: $\alpha_r > 2$. Choose $s = r$, $\beta_j = \alpha_j$ for $0 \leqq j \leqq r - 2$, $\beta_{r-1} = \alpha_{r-1} + 2$, $\beta_r = \alpha_r - 1$. Q.E.D.

By use of Lemma 10, we see that the conclusion of Lemma 9 is equivalent to the conclusion of Lemma 4. This sufficies for the proof of the part of Theorem I concerning integral Stiefel-Whitney classes.

The proof of Theorem III bears the same relation to Lemma 9 that the proof of Theorem II bears to Lemma 4. The details of the proof of conclusions (a), (b), (c), and (d) of Theorem III are completely analogous to the proof of the corresponding conclusions of Theorem II. There is no point in repeating these details here; the reader can easily work them out himself.

Only conclusion (e) involves anything new. The fact that $\alpha_r > 1$ is proved in Lemma 9. It remains to prove that $\alpha_r < k + 1$. If $\alpha_r = k + 1$, then $\alpha_0 = \alpha_1 = \cdots = \alpha_{r-1} = 0$ since $\sum \alpha_j = k + 1$. Now apply Lemma 3 with $j = r$; the result is

$$\mathrm{Sq}^I x = x^{2^r} \in H^n(M^n, \mathbf{Z}_2),$$

since $J = (0)$ in this case. However, x is the reduction mod 2 of an integral cohomology class of finite order; i.e.,

$$x = \rho(y)$$

where $y \in T^{k+1}(M^n, \mathbf{Z})$. Therefore

$$x^{2^r} = \rho(y^{2^r});$$

but $y^{2^r} = 0$ since $H^n(M^n, \mathbf{Z})$ is an infinite cyclic group and y is a torsion element. This contradicts the fact that $\mathrm{Sq}^I x \neq 0$.

This completes the proof of Theorem III.

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