ON TORSIONFREE, TORSION AND PRIMARY SPECTRA

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1. Introduction

In this note we will show that the following two theorems on abelian groups have analogues for spectra:

I. Every abelian group is an extension of its torsion subgroup by a torsion-free group.

II. Every (abelian) torsion group is the direct sum of its primary components.

The paper is written in semisimplicial language and we shall freely use the results on semisimplicial spectra of [3].

2. The group hom (X, Y)

With every two spectra X and Y we will associate an abelian group hom (X, Y) whose elements are, roughly speaking, the homotopy classes of the maps of X into Y.

Definition 2.1. Let $X, Y \in \mathcal{S}_p$ and let G denote the set of all maps $X \to FY \in \mathcal{S}_p$. Then G may be turned into a group by defining $(ab)\sigma = (a\sigma)(b\sigma)$ for all $a, b \in G$ and $\sigma \in X$. The subset $H \subset G$ consisting of the maps which are homotopic to the constant map ($\sigma \to *$ for all $\sigma \in X$) is readily verified to be a normal subgroup, and we thus may define

$$\text{hom } (X, Y) = G/H.$$

As, for $a, b \in G$, we have $a \sim b$ if and only if $ab^{-1} \in H$, the elements of hom (X, Y) are exactly the homotopy classes of maps $X \to FY \in \mathcal{S}_p$. Moreover if $Y \in \mathcal{S}_{pE}$, then ([3], 5.3 and 9.2) the map $fY : Y \to FY$ induces a one-to-one correspondence between the elements of hom (X, Y) and the homotopy classes of maps $X \to Y \in \mathcal{S}_p$.

Any two maps $v : X' \to X$, $w : Y \to Y' \in \mathcal{S}_p$ induce a homomorphism hom (v, w) : hom $(X, Y) \to$ hom (X', Y') and the function hom (,) so defined clearly is a functor contravariant in the first variable and covariant in the second.

Example 2.2. Let $X \in \mathcal{S}_p$ and let S^q denote the spectrum which has a simplex Ψ of degree q as its only non-degenerate simplex. Then it is easily seen that the function which assigns to every map $a : S^q \to FX$ the simplex $a\Psi \in FX$ induces an isomorphism,

$$\text{hom } (S^q, X) \approx \pi_q X.$$

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PROPOSITION 2.3. *Let $A, X \in \mathcal{S}_P$; let $B \subset A$ be a subspectrum; let $A \setminus B$ be the spectrum obtained from A by identifying every simplex of B with the appropriate base point; and let $j : B \to A$ be the inclusion and $p : A \to A \setminus B$ the identification map. Then the following sequence is exact:*

$$\hom(A\setminus B, X) \xrightarrow{\hom(p,\, i_X)} \hom(A, X) \xrightarrow{\hom(j,\, i_X)} \hom(B, X).$$

Proof. This follows at once from the homotopy extension theorem for spectra which is proved just as the one for set complexes ([2]).

PROPOSITION 2.4. *If $v : X \to X'$ is a weak homotopy equivalence and $Y \in \mathcal{S}_P$, then* hom (v, i_Y) *is an isomorphism.*

Proof. The proposition is clearly true if v is a homotopy equivalence. Hence it sufficies to consider the case $v = fX : X \to FX$. That hom (fX, i_Y) is an epimorphism follows from the fact that for every map $a : X \to FY$ there is a (unique) homomorphism $a' : FX \to FY$ such that $a'(fX) = a$. Moreover, for any map $a'' : FX \to FY$ with $a''(fX) = a$, we have $(Fa'')(FfX) = Fa = (Fa')(FfX)$. This implies $Fa'' \approx Fa'$ and hence $a'' \sim a'$; i.e., hom (fX, i_Y) is also a monomorphism.

PROPOSITION 2.5. hom (X, Y) *is abelian for all $X, Y \in \mathcal{S}_P$.*

Proof. The two projections $X \vee X \to X$ induces homomorphisms hom $(X, Y) \to$ hom $(X \vee X, Y)$ and hence a homomorphism $j :$ hom $(X, Y) \times$ hom $(X, Y) \to$ hom $(X \vee X, Y)$ which is clearly an isomorphism. By the argument of the proof of 5.3 of [3], one shows that the inclusion map $X \vee X \to X \times X$ is a weak homotopy equivalence. Hence (2.4) it induces an isomorphism $k :$ hom $(X \times X, Y) \to$ hom $(X \vee X, Y)$. Finally, if $d :$ hom $(X \times X, Y) \to$ hom (X, Y) is the map induced by the diagonal map $X \to X \times X$, then a simple computation yields that the composite *homomorphism*,

$$dk^{-1}j : \hom (X, Y) \times \hom (X, Y) \to \hom (X, Y),$$

is such that $dk^{-1}j(\alpha, 1) = \alpha = dk^{-1}j(1, \alpha)$ for all $\alpha \in$ hom (X, Y). Hence hom (X, Y) is abelian.

3. Homotopy extensions

DEFINITION 3.1. Let $V, W, X \in \mathcal{S}_{PE}$. Then X is said to be a *homotopy extension* of V by W if there exists a fibre map ([4], §3) $a : X' \to W$ such that (i) V is the fibre of a and (ii) X' has the same homotopy type as X.

Example 3.2. Let $b : X \to Y \in \mathcal{S}_{PE}$ with $Y \in \mathcal{S}_{PG}$ and denote by ϕb the spectrum of which a simplex of degree q is any pair (σ, τ) with $\sigma \in X_{(q)}$, $\tau \in \wedge Y$ ([4], §2) and $b\sigma = \delta_0\tau$; its faces and degeneracies are given by $d_i(\sigma, \tau) = (d_i\sigma, d_i\tau)$ and $s_i(\sigma, \tau) = (s_i\sigma, s_i\tau)$ for all i. Then *X is a homotopy extension of ϕb by Y.* This can be seen as follows. Let $c : X \times \wedge Y \to Y$ be the map given by $c(\sigma, \tau) = (b\sigma)(\delta_0\tau)^{-1}$ then a simple calculation yields that c is a fibre map. Clearly ϕb

is its fibre and the contractibility of $\wedge Y$ ([4], §2) implies that X and $X \times \wedge Y$ have the same homotopy type.

4. Classes of spectra

With every collection of abelian groups we will associate a class of spectra.

DEFINITION 4.1. Call a spectrum *finite* if it has only a finite number of non-degenerate simplices. If C is a collection of abelian groups, then a spectrum X will be called a *C-spectrum* if hom $(W, X) \in C$ for all finite W. So in this sense we can talk of torsion spectra, torsionfree spectra, primary spectra, divisible spectra, etc.

An immediate consequence of 2.2 is

PROPOSITION 4.2. *If X is a C-spectrum, then $\pi_q X \in C$ for all q.*

The converse of this proposition need not be true. In fact

PROPOSITION 4.3. *A spectrum X is torsionfree if and only if $\pi_q X$ is torsionfree and divisible for all q.*

Proof. Suppose X is torsionfree. Let A be a spectrum with only three nondegenerate simplices α, β and γ with degree $\alpha = q - 1, d_0\beta = d_{2j}\gamma = \alpha$ for $0 \leqq j < n$ and $d_i\beta = *$, $d_i\gamma = *$ otherwise and let $B \subset A$ be the subspectrum generated by α and γ. Then clearly the map $A \rightarrow S^q$ given by $\beta \rightarrow *$, $\gamma \rightarrow \Psi$ and the map $A \diagdown B \rightarrow S^q$ given by $\beta \rightarrow \Psi$ are weak homotopy equivalences (and therefore (2.2 and 2.4) hom (A, X) and hom $(A \diagdown B, X)$ are isomorphic to $\pi_q X$) and the map induced by the projection $A \rightarrow A \diagdown B$ is "multiplication by n." The exactness (2.3) of the sequence hom $(A \diagdown B, X) \rightarrow$ hom $(A, X) \rightarrow$ hom (B, X) then implies that hom (B, X) contains a subgroup isomorphic to $\pi_q X \otimes Z_n$. As hom (B, X) is torsionfree, this means that $\pi_q X \otimes Z_n = 0$ for all n; i.e., $\pi_q X$ is divisible.

The converse is an immediate consequence of

PROPOSITION 4.4. *Let C be a class of abelian groups in the sense of Serre* ([5]). *Then X is a C-spectrum if and only if $\pi_q X \in C$ for all q.*

Proof. If $W \in \mathcal{S}_P$ has only one non-degenerate simplex, then (2.2) hom $(W, X) \in C$. Now suppose it has already been proved that hom $(W, X) \in C$ if W has less than n non-degenerate simplices. If $V \in \mathcal{S}_P$ has n non-degenerate simplices, let $W \subset V$ be a subspectrum with $n - 1$ non-degenerate simplices. The exactness of the sequence (2.3) hom $(V \diagdown W, X) \rightarrow$ hom $(V, X) \rightarrow$ hom (W, X) then clearly implies that hom $(V, X) \in C$.

5. The torsionfree part of a spectrum

With every spectrum in $\mathcal{S}_{PE}$ we shall now associate (in a natural manner) a torsionfree spectrum, called its torsionfree part. First we define

DEFINITION 5.1. If $\mathcal{G}$ denotes the category of groups, then a covariant functor $\otimes : \mathcal{S}_{\mathcal{P}}, \mathcal{G} \to \mathcal{S}_{\mathcal{P}G}$ may be defined as follows. For $X \in \mathcal{S}_{\mathcal{P}}$ and $B \in \mathcal{G}$, $(X \otimes B)_{(q)}$ is the group with

(i) a generator $\sigma \otimes \beta$ for every $\sigma \in X_{(q)}$ and $\beta \in B$,

(ii) a relation $* \otimes \beta = *$ for all $\beta \in B$,

(iii) a relation $(\sigma \otimes \beta)(\sigma \otimes \beta') = (\sigma \otimes \beta\beta')$ for every $\sigma \in X_{(q)}$ and $\beta, \beta' \in B$. The operators in $X \otimes B$ are given by the formulas

$$d_i(\sigma \otimes \beta) = d_i\sigma \otimes \beta, \qquad s_i(\sigma \otimes \beta) = s_i\sigma \otimes \beta, \quad \text{for all } i.$$

Similarly for maps $w : X \to Y \in \mathcal{S}_{\mathcal{P}}$ and $a : B \to C \in \mathcal{G}$ the homomorphism $w \otimes a : X \otimes B \to Y \otimes C$ is the one given by $\sigma \otimes \beta \to w\sigma \otimes a\beta$ for all $\sigma \in X$ and $\beta \in B$.

Example 5.2. Let Z denote the additive group of the integers and for every $X \in \mathcal{S}_{\mathcal{P}}$ and $\sigma \in X$ identify $\sigma \otimes 1$ with $F\sigma \in FX$. Then the functor $\otimes Z : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}G}$ clearly coincides with the functor F.

DEFINITION 5.3. Let Q denote the additive group of the rationals. For $X \in \mathcal{S}_{\mathcal{P}E}$ the spectrum $X \otimes Q$ will be called the *torsionfree part* of X and we shall denote by $j : X \to X \otimes Q$ the map given by $\sigma \to \sigma \otimes 1$ for all $\sigma \in X$. Then we have

PROPOSITION 5.4. *Let $X \in \mathcal{S}_{\mathcal{P}E}$. Then $X \otimes Q$ is torsionfree. In fact, for all q,*

$$\pi_q(X \otimes Q) = \pi_q X \otimes Q \quad \textit{and} \quad (\pi_q j)\alpha = \alpha \otimes 1 \quad \textit{for all} \quad \alpha \in \pi_q X.$$

Proof. For any abelian group A, $A \otimes Q$ is the direct limit of the sequence

$$A \xrightarrow{2} A \xrightarrow{3} \cdots \xrightarrow{n} A \xrightarrow{n+1} \cdots$$

where n denotes "multiplication by n." Similarly $X \otimes Q$ is the direct limit of the sequence

$$X \xrightarrow{fX} FX = X \otimes Z \xrightarrow{i_X \otimes 2} \cdots \xrightarrow{i_X \otimes n} X \otimes Z \xrightarrow{i_X \otimes (n+1)} \cdots$$

Clearly $\pi_q(i_X \otimes n)\alpha = n\alpha$ for all $\alpha \in \pi_q(X \otimes Z)$. The proposition now follows from the fact that the homotopy groups commute with direct limits.

COROLLARY 5.5. *A spectrum $X \in \mathcal{S}_{\mathcal{P}E}$ is torsionfree if and only if it has the same homotopy type as its torsionfree part.*

COROLLARY 5.6. *If two spectra $X, Y \in \mathcal{S}_{\mathcal{P}E}$ have the same homotopy type, then so do their torsionfree parts.*

Remark 5.7. A different way of obtaining the torsionfree part of a spectrum is due to E. H. Brown, Jr. It uses his representability theorem for generalized cohomology theories ([1]). If Y is a spectrum, H the corresponding cohomology theory, H_Q the cohomology theory obtained from H by tensoring with Q and Y_Q a spectrum corresponding with H_Q, then (up to homotopy) Y_Q is the torsion free part of Y.

6. The torsion part of a spectrum

With every spectrum $X \in \mathcal{S}_{pE}$ we associate (in a natural manner) a torsion spectrum, called its torsion part. Then X is a homotopy extension of its torsion part by its torsionfree part.

DEFINITION 6.1. Let $X \in \mathcal{S}_{pE}$. Then the *torsion part* of X is the spectrum $TX = \phi j$ where ϕ is as in §3 and $j : X \to X \otimes Q$ as in §5. Example 3.2 implies

PROPOSITION 6.2. *Every spectrum in $\mathcal{S}_{pE}$ is a homotopy extension of its torsion part by its torsionfree part.*

PROPOSITION 6.3. *Let $X \in \mathcal{S}_{pE}$. Then TX is a torsion spectrum. In fact there is a split exact sequence*

$$0 \to \pi_{q+1}X \otimes Q/Z \to \pi_q TX \to \mathrm{Tor}\,(\pi_q X,\, Q/Z) \to 0.$$

Proof. Combination of the exact homotopy sequence ([4], §3) of the fibre map $X \times \wedge(X \otimes Q) \to X \otimes Q$ of 3.2 yields the desired short exact sequence. That it splits follows from the fact that Q/Z and hence $\pi_{q+1}X \otimes Q/Z$ is divisible.

COROLLARY 6.4. *A spectrum $X \in \mathcal{S}_{pE}$ is a torsion spectrum if and only if it has the same homotopy type as its torsion part.*

COROLLARY 6.5. *If two spectra X, $Y \in \mathcal{S}_{pE}$ have the same homotopy type, then so do their torsion parts.*

Also

PROPOSITION 6.6. *Let V, W, $X \in \mathcal{S}_{pE}$ be such that X is a homotopy extension of V by W and let V be a torsion spectrum and W a torsionfree one. Then V has the same homotopy type as TX and W the same as $X \otimes Q$.*

Proof. Let $a : X' \to W \in \mathcal{S}_{pE}$ be as in 3.1. By 5.4, $a \otimes i_Q : X' \otimes Q \to W \otimes Q$ induces an isomorphism of all homotopy groups; by 5.6., $X \otimes Q$ and $X' \otimes Q$ have the same homotopy type and, by 5.5, so do W and $W \otimes Q$. Hence W and $X \otimes Q$ have the same homotopy type.

Now let $b : W \to X' \otimes Q$ be a homotopy equivalence such that $ba \sim j : X' \to X' \otimes Q$, let $d : X' \to X' \times \wedge(X' \otimes Q)$ be given by $\sigma \to (\sigma, *)$ for all $\sigma \in X'$, and let $c : X' \times \wedge(X' \otimes Q) \to X' \otimes Q$ be as in 3.2. Then $cd = j$. As c is a fibre map, the homotopy lifting theorem for spectra (which is proved just as the one for set complexes [2]) yields a map $d' : X' \to X' \times \wedge(X' \otimes Q)$ such that $cd' = ba$. As b and d' are homotopy equivalences, so is the restriction $d'/V : V \to TX'$ and hence (6.5) V and TX have the same homotopy type.

7. Primary spectra

With every prime p and spectrum $X \in \mathcal{S}_{pE}$ we shall associate (in a natural manner) a p-primary spectrum, called its p-primary part. Then the (weak) product of the p-primary parts of X has the homotopy type of its torsion part.

DEFINITION 7.1. Let p be a prime, let Q_p denote the additive group of the rationals of the form q/p^n, let $X \in \mathcal{S}_{pE}$, and let $j_p : X \to X \otimes Q_p$ be the map given by $\sigma \to \sigma \otimes 1$ for all $\sigma \in X$. The *p-primary part* of X then is the spectrum $T_pX = \phi j_p$, where ϕ is as in 3.2.

PROPOSITION 7.2. *Let $X \in \mathcal{S}_{pE}$. Then T_pX is a p-primary spectrum. In fact there is a split exact sequence*

$$0 \to \pi_{q+1}X \otimes Q_p/Z \to \pi_q T_p X \to \text{Tor}\ (\pi_q X, Q_p/Z) \to 0.$$

The proof is similar to that of 5.4 and 6.3, using the fact that for any abelian group A, $A \otimes Q_p$ is the direct limit of the sequence

$$A \xrightarrow{p} A \xrightarrow{p} \cdots \xrightarrow{p} A \xrightarrow{p} \cdots$$

COROLLARY 7.3. *A spectrum $X \in \mathcal{S}_{pE}$ is a p-primary spectrum if and only if it has the same homotopy type as its p-primary part.*

COROLLARY 7.4. *If two spectra $X, Y \in \mathcal{S}_{pE}$ have the same homotopy type, then so do their p-primary parts.*

PROPOSITION 7.5. *Let $X \in \mathcal{S}_{pE}$ and let W be the weak product of the spectra T_pX. Then W has the same homotopy type as TX.*

Proof. Let V be the union of the T_pX and let $a : V \to W$ be the inclusion map. Then one shows as in the proof of 5.3 of [3], that u is a weak homotopy equivalence. The inclusions $Q_p \to Q$ induce maps $X \otimes Q_p \to X \otimes Q$ and hence maps $t_p : T_pX \to TX$, which clearly map the homotopy groups of T_pX isomorphically onto the p-primary component of the homotopy groups of TX. This implies that the map $t : V \to TX$ induced by the maps t_p induces an isomorphism of all homotopy groups. Hence W and TX have the same homotopy type.

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