ON WU'S FORMULA OF STEENROD SQUARES ON STIEFEL-WHITNEY CLASSES*

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Introduction

In this paper, we shall rederive Wu's formula of Steenrod squares on Stiefel-Whitney classes of a vector bundle ([1], [3], [7]) from Adem's relations ([2], [5]). Wu originally proved the formula for the squares in Grassmannian mainfolds which are classifying spaces[1] for orthogonal groups; therefore, it is only valid for vector bundles (or sphere bundles). Since our proof is based on Adem's relations and some elementary binomial coefficient identities, we actually show that the formula holds for those fibre spaces whose cohomological behavior is like a vector bundle. Throughout this paper we shall use only the ring of integers mod 2, Z_2, for coefficients of any cohomology group or cohomology class. This paper was inspired by a conversation with Professor Milnor, to whom the author extends his thanks. He also wishes to express his gratitude to Professor Steenrod for communicating to him Lemma 1, which simplifies a great deal of the original proof.

1. Thom's isomorphism ϕ

Let $p : E \rightarrow B$ be a fibration in the sense of Serre with arc-wise connected base space B and fibre $F = p^{-1}(b)$, $b \in B$. Let E_0 be a subspace of the total space of E of the fibration $p : E \rightarrow B$ such that $(p \mid E_0) : E_0 \rightarrow B$ is also a fibration with fibre $F_0 = (p \mid E_0)^{-1}(b)$, $b \in B$. Assume $H^i(F, F_0) = 0$, $i \neq n$ and $H^n(F, F_0) = Z_2$. Let $j = (F, F_0) \rightarrow (E, E_0)$ be the inclusion map inducing $j^* : H^*(E, E_0) \rightarrow H^*(F, F_0)$, and let U_b be the generator of $H^n(F, F_0)$. Cockcroft ([4]) proved the following result, which is a generalization of Thom's isomorphism theorem of ([6]).

$H^k(E, E_0) = 0$ for $k < n$, and, for any $b \in B$, there exists a unique class $U \in H^n(E, E_0)$ such that $j^*U = U_b$ and, moreover, such that if the projection $p : E \rightarrow B$ induces $p^* : H^*(B) \rightarrow H^*(E)$, then the correspondence $x \rightarrow U \smile p^*x$ defined for all $x \in H^*(B)$, defines an isomorphism $\phi : H^i(B) \approx H^{n+i}(E, E_0)$. Following Thom ([6]), we define

$$W_j = \phi^{-1} \operatorname{Sq}^j U \tag{1}$$

and call $W_j \in H^j(B)$ the jth Stiefel-Whitney class of the fibration $p : E \rightarrow B$.

2. Wu's formula

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[1] Actually, the finite Grassmannian manifold $G_{m,n}$ is only the classifying space for complexes of dimension less than m. But, the infinite Grassmannian manifold G_n which is the limit of $G_{m,n}$ as m tends to infinity is the classifying space for any paracompact space.

Before we state the theorem of squares on W_j, we clarify the convention of the symbol $\binom{a}{b}$.

$$(2)\qquad \begin{aligned} \binom{a}{b} &= \text{binomial coefficient} && \text{Mod 2 if } a \geqq b \geqq 0, \\ &= 1 && \text{Mod 2 if } a = -1 \text{ and } b = 0, \\ &= 0 && \text{Mod 2 otherwise.} \end{aligned}$$

Let $p\colon E \to B$ be a fibration satisfying the condition in §1 and $\{W_j\}$, the Stiefel-Whitney classes of the fibration. We prove the following result which is Wu's formula when $p\colon E \to B$ is a vector bundle of dimension n.[2]

THEOREM. *We have the following formula:*

$$(3)\qquad \mathrm{Sq}^a\, W_b = \sum_{0 \leqq t \leqq a} \binom{b - a + t - 1}{t} W_{a-t} \smile W_{b+t} \qquad \text{Mod 2}$$

The proof of the theorem will be given in §4. We remark that if $a > b$, then the left hand side is automatically zero and the right hand side of (3) is zero because of the convention (2).

3. Arithmetic lemmas

For the proof of the theorem, we need the following elementary arithmetic lemmas on binomial identities.

LEMMA 1.[3] *If* $\lambda \geqq \mu \geqq 0$ *and* $\rho > 0$ *are integers, then*

$$(4)\qquad \binom{\lambda}{\mu} + \binom{\lambda + \rho}{\mu} = \sum_{\rho \leqq 2\nu \leqq \min(2\rho, \mu+\rho)} \binom{\lambda + \rho - \nu}{\mu + \rho - 2\nu} \binom{\nu - 1}{\rho - \nu} \qquad \text{Mod 2}$$

Proof. Let $P(\lambda, \mu, \rho)$ denote the formula (4) which depends on three variables and let $Q(\rho)$ be the union of the formulas $P(\lambda, \mu, \zeta)$ for all $\lambda \geqq \mu \geqq 0$, $\zeta \leqq \rho$. Now, $Q(1)$ is just a Mod 2 version of the formula

$$(5)\qquad \binom{\lambda + 1}{\mu} = \binom{\lambda}{\mu} + \binom{\lambda}{\mu - 1},$$

which is well known. We claim that $Q(2)$ is also true. For this purpose, it suffices to show that $P(\lambda, \mu, 2)$ is also true. Replacing λ by $\lambda + 1$ in (5), we have

$$(6)\qquad \binom{\lambda + 2}{\mu} = \binom{\lambda + 1}{\mu} + \binom{\lambda + 1}{\mu - 1}.$$

Applying $Q(1)$ twice to (6) reduced to Mod 2, we have

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[2] The dimension of the fibre of a vector bundle is called the dimension of the vector bundle.

[3] Due to Steenrod.

$$(7)\qquad \begin{aligned}\binom{\lambda+2}{\mu} &= \binom{\lambda}{\mu} + \binom{\lambda}{\mu-1} + \binom{\lambda+1}{\mu-1} \\ &= \binom{\lambda}{\mu} + \binom{\lambda}{\mu-2}\end{aligned} \qquad \text{Mod } 2$$

which is $P(\lambda, \mu, 2)$. Now, suppose that $Q(\rho - 1)$ is true for $\rho > 2$. Both $P(\lambda + 1, \mu, \rho - 2)$ and $P(\lambda, \mu - 1, \rho - 1)$ are true by $Q(\rho - 1)$. We write the formula $P(\lambda + 1, \mu, \rho - 2)$ and write directly under it the formula $P(\lambda, \mu - 1, \rho - 1)$; i.e.,

$$(8)\qquad \begin{aligned}&\binom{\lambda+1}{\mu} + \binom{\lambda+\rho-1}{\mu} \\ &\quad = \sum_{(\rho-2) \leqq 2\nu \leqq \min(2(\rho-2), \mu+\rho-2)} \binom{\lambda+\rho-1-\nu}{\mu+\rho-2-2\nu}\binom{\nu-1}{\rho-2-\nu} \\ &\binom{\lambda}{\mu-1} + \binom{\lambda+\rho-1}{\mu-1} \\ &\quad = \sum_{(\rho-1) \leqq 2\nu \leqq \min(2(\rho-1), \mu+\rho-2)} \binom{\lambda+\rho-1-\nu}{\mu+\rho-2-2\nu}\binom{\nu-1}{\rho-1-\nu}\end{aligned} \qquad \text{Mod } 2.$$

Now, add the two formulas in (8) by adding corresponding terms. In each column of the addition we may apply $Q(1)$ to reduce two terms to one. The resulting formula is just $P(\lambda, \mu, \rho)$. Since $\lambda \geqq \mu \geqq 0$ are arbitrary, this proves that $Q(\rho - 1)$ implies $Q(\rho)$ and the proof of this lemma is complete by induction.

LEMMA 2.

$$(9)\qquad \binom{b-v-1}{a-u-v} + \binom{b-u-1}{a-u-v} + \sum_{0 \leqq i \leqq [a/2]} \binom{b-i-1}{a-2i}\binom{i-u-1}{v-1} = 0 \qquad \text{Mod } 2$$

for $0 < u < v$ *and* $u + v \leqq a \leqq b$.

This lemma is a special case of Lemma 1 by letting $\lambda = b - u - 1$, $\mu = a - u - v$ and $\rho = v - u$.

LEMMA 3.

$$(10)\qquad \binom{b-1}{a-u} + \sum_{0 \leqq i \leqq [a/2]} \binom{b-i-1}{a-2i}\binom{i-1}{u-i} = \binom{b-a+(a-u)-1}{a-u} \qquad \text{Mod } 2$$

for $0 < u \leqq [a/2]$ and $a \leqq b$.

This lemma is again a special case of Lemma 1 by letting $\lambda = b - u - 1$, $\mu = a - u$ and $\rho = u$.

4. Proof of Wu's Formula

Now, we are ready to prove the theorem. First, we order the set of pairs of non-negative integers $\{(a, b)\}$ as follows. We say $(a_1, b_1) \leqq (a_2, b_2)$ if $b_1 < b_2$

or $b_1 = b_2$, $a_1 \leqq a_2$. To each pair (a, b), there corresponds $\mathrm{Sq}^a W_b$. We prove our Theorem by induction on (a, b) under this ordering. We claim that $\mathrm{Sq}^1 W_b$ which corresponds to $(1, b)$ satisfies (3). In fact, it follows from the definition of W_b and Cartan formula of squares ([5]) that

$$\begin{aligned} \mathrm{Sq}^1\mathrm{Sq}^b U &= \mathrm{Sq}^1(U \smile p^*W_b) \\ &= \mathrm{Sq}^1 U \smile p^*W_b + U \smile p^*\mathrm{Sq}^1 W_b \\ &= U \smile p^*(W_1 \smile W_b + \mathrm{Sq}^1 W_b). \end{aligned} \tag{11}$$

Since

$$\begin{aligned} \mathrm{Sq}^1\mathrm{Sq}^b &= \mathrm{Sq}^{b+1}, \quad \text{for } b \text{ even}, \\ &= 0, \quad \text{for } b \text{ odd}, \end{aligned} \tag{12}$$

the verification of (3) for $\mathrm{Sq}^1 W_b$ follows immediately from (11), (12) and Thom's isomorphism ϕ.

Suppose that $\mathrm{Sq}^a W_b$ satisfies (3) for $(a, b) \leqq (i-1, j)$ such that $i - 1 \geqq 1$. Once we prove that $\mathrm{Sq}^i W_j$ satisfies (3), the Theorem follows from induction. As we remarked in §2, if $i > j$, then both side of (3) are zero. Hence, (3) is verified. Therefore, we assume $i \leqq j$. By the definition of W_j and Cartan formula of squares, we have

$$\begin{aligned} \mathrm{Sq}^i\mathrm{Sq}^j U &= \mathrm{Sq}^i(U \smile p^*W_j) \\ &= \textstyle\sum_{0<n\leqq i} \mathrm{Sq}^n U \smile p^*\mathrm{Sq}^{i-n}W_j \\ &= \textstyle\sum_{0<n\leqq i} \mathrm{Sq}^n U \smile p^*\mathrm{Sq}^{i-n}W_j + U \smile p^*\mathrm{Sq}^i W_j. \end{aligned} \tag{13}$$

Since $i \leqq j$, it follows from Adem's relations ([2], [5]) that

$$\mathrm{Sq}^i\,\mathrm{Sq}^j = \sum_{0\leqq m\leqq[i/2]} \binom{j-m-1}{i-2m} \mathrm{Sq}^{i+j-m}\,\mathrm{Sq}^m. \tag{14}$$

Substituting (14) into (13) and using Cartan formula again, we have

$$\begin{aligned} &U \smile p^*\,\mathrm{Sq}^i\,W_j \\ &= \sum_{0<n\leqq i} U \smile p^*(W_n \smile \mathrm{Sq}^{i-n}W_j) \\ &\qquad + \sum_{0\leqq m\leqq[i/2]} \binom{j-m-1}{i-2m} \mathrm{Sq}^{i+j-m}(U \smile p^*W_m) \\ &= \sum_{0<n\leqq i} U \smile p^*(W_n \smile \mathrm{Sq}^{i-n}\,W_j) \\ &\qquad + \sum_{\substack{0\leqq m\leqq[i/2] \\ 0\leqq m+t\leqq 2[i/2]}} \binom{j-m-1}{i-2m} (\mathrm{Sq}^{i+j-m-t} \smile p^*\,\mathrm{Sq}^t\,W_m) \\ &= U \smile p^*\Bigg\{\sum_{0\leqq n\leqq i} W_m \smile \mathrm{Sq}^{i-n}\,W_j \\ &\qquad + \sum_{\substack{0\leqq m\leqq[i/2] \\ 0\leqq m+t\leqq 2[i/2]}} \binom{j-m-1}{i-2m} W_{i+j-m-t} \smile \mathrm{Sq}^t\,W_m\Bigg\} \end{aligned} \tag{15}$$

Applying ϕ^{-1} to (15), we have

$$\begin{aligned}\mathrm{Sq}^i W_j &= \sum_{0<n\leqq i} W_n \smile \mathrm{Sq}^{i-n} W_j \\ &\quad + \sum_{\substack{0\leqq m\leqq [i/2] \\ 0\leqq m+t\leqq 2[i/2]}} \binom{j-m-1}{i-2m} W_{i+j-m-t} \smile \mathrm{Sq}^t W_m .\end{aligned} \tag{16}$$

Using the induction hypothesis, the right hand side of (16) is the sum of the following four types of terms with the coefficients given right behind them

(A) $W_u \smile W_v \smile W_{i+j-u-v} \quad (0 < u < v < i, u + v \leqq i)$

$$\begin{aligned}&\binom{j-i+u+i-u-v-1}{i-u-v} + \binom{j-i+v+i-u-v-1}{i-u-v} \\ &+ \sum_{[(u+v+1)/2]\leqq n\leqq \min([i/2],v)} \binom{j-n-1}{i-2n}\binom{n-u-1}{v-n} \\ &= \binom{j-v-1}{1-u-v} + \binom{j-u-1}{i-u-v} \\ &\qquad + \sum_{[(u+v+1)/2]\leqq n\leqq \min([i/2],v)} \binom{j-n-1}{i-2n}\binom{n-u-1}{v-n}\end{aligned} \tag{17}$$

(B) $W_u^2 \smile W_{i+j-2u} \quad (0 < u \leqq [i/2])$

$$\begin{aligned}&\binom{j-i+u-i-2u-1}{i-2u} + \binom{j-u-1}{i-2u}\binom{u-u-1}{0} \\ &= \binom{j-u-1}{i-2u} + \binom{j-u-1}{i-2u} = 0\end{aligned} \tag{18}$$

(C) $W_u \smile W_{i+j-u} \quad (0 < u \leqq i)$

$$\begin{aligned}&\binom{j-i+u+i-u-1}{i-u} \\ &\qquad + \sum_{[(u+1)/2]\leqq n\leqq \min([i/2],u)} \binom{j-n-1}{i-2n}\binom{n-1}{u-n} \\ &= \binom{j-1}{i-u} + \sum_{[(u+1)/2]\leqq n\leqq \min([i/2],u)} \binom{j-n-1}{i-2n}\binom{n-1}{u-n}\end{aligned} \tag{19}$$

(D) W_{i+j}

$$\binom{j-i-1}{i} \tag{20}$$

By the convention (2), we have

$$\sum_{[(u+v+1)/2] \leqq n \leqq \min([i/2],v)} \binom{j-n-1}{i-2n}\binom{n-u-1}{v-n} = \sum_{0 \leqq n \leqq [i/2]} \binom{j-n-1}{i-2n}\binom{n-u-1}{v-n}, \tag{21}$$

$$\sum_{[(u+v+1)/2] \leqq n \leqq \min([i/2],v)} \binom{j-n-1}{i-2n}\binom{n-1}{u-n} = \sum_{0 \leqq n \leqq [i/2]} \binom{j-n-1}{i-2n}\binom{n-1}{u-n}. \tag{22}$$

Substituting (21), (22) into (17), (19) and applying Lemma 2 and Lemma 3 to them respectively, we have that the coefficient of $W_u \smile W_v \smile W_{i+j-u-v}$ $(0 < u < v < i, u + v \leqq i)$ is zero and the coefficient of $W_u \smile W_{i+j-u} (0 < u \leqq i)$ is

$$\binom{j-i+i-u-1}{i-u}. \tag{23}$$

Hence, $\mathrm{Sq}^i W_j$ satisfies (3) and the proof of the theorem is complete.

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