ON DOMAINS OF CONTROLLABILITY OF PROPER AND NORMAL SYSTEMS

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Introduction

Consider the linear control system

$$\dot{x} = b_1 u(t)$$
$$\dot{y} = x,$$

where $b_1 > 0$, $u(t) \in L$ (Lebesgue measurable in $\langle 0, \infty))$, $|u(t)| \leq 1$. Denote by S_T the set of all points in E_2 that can be reached from the origin in a time $T > 0$ with some $u(t)$ satisfying the above conditions; i.e. $S_T = E(z \in E_2$; $z = e^{AT} \int_0^T e^{-A\tau} \begin{pmatrix} b_1 \\ 0 \end{pmatrix} u(\tau)\, d\tau,\ u \in L,\ |u| \leq 1)$ where $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Since

$$e^{AT} \int_0^T e^{-A\tau} \begin{pmatrix} b_1 \\ 0 \end{pmatrix} u(\tau)\, d\tau = e^{AT} T \int_0^1 e^{-AT\sigma} \begin{pmatrix} b_1 \\ 0 \end{pmatrix} u(T\sigma)\, d\sigma$$
$$= \begin{pmatrix} T & 0 \\ 0 & T^2 \end{pmatrix} \int_0^1 \begin{pmatrix} b_1 \\ -\sigma b_1 \end{pmatrix} u(T\sigma)\, d\sigma,$$

each point $z_T \in S_T$ may be written as $z_T = \begin{pmatrix} T & 0 \\ 0 & T^2 \end{pmatrix} \tilde{z}$ where

$$\tilde{z} \in \tilde{\mathfrak{a}},\ \tilde{\mathfrak{a}} = E\left(w \in E_2\, ;\, w = \int_0^1 \begin{pmatrix} b_1 \\ -\sigma b_1 \end{pmatrix} u(\sigma)\, d\sigma, \quad u \in L, |u| \leqq 1\right).$$

Thus in this simple case S_T for any $T > 0$ can be calculated as a linear transformation of the set $\tilde{\mathfrak{a}}$ independent from T. In the present paper it is shown that this circumstance holds asymptotically for $T \to \infty$ for any proper system with zero real parts in the characteristic roots.

To further clarify the results obtained afterwards (Section 2), consider the control system

$$\dot{x} = Ax + Bu,$$

where $x = \text{col}(x_1, \cdots, x_n)$, A has only ones below the main diagonal, $B = (b_{ij})$ $i = 1, 2, \cdots, n$, $j = 1, \cdots, s$, and $u = \text{col}(u_1, \cdots, u_s)$. Now if $\mathfrak{a}_T$ is defined as the set given by

$$\mathfrak{a}_T = E\left(x; x = \int_0^T e^{-A\sigma} Bu(\sigma)\, d\sigma; \quad \text{for all admissible } u\right),$$

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then by first making a change of variable $\sigma = T\tau$ and arranging things properly one obtains

$$\mathfrak{A}_T = E\left(x; x = \sum_{i=1}^{n} \int_0^1 T_i\, Q_i(\tau) B_i\, u(\tau)\, d\tau; \text{ for all admissible } u\right),$$

where T_i, Q_i are $n \times n$ matrices, B_i are $n \times s$ matrices; $T_i = \text{diag}(0, \cdots, 0, T, T^2, \cdots, T^{n-i+1})$, T appearing in the ith place; $Q_i(\tau)$ has all columns zero except the ith which is the same as that of $e^{-A\tau}$; and B_i has all rows zero except the ith, this being the same as in B. From this, two things are immediately apparent. First we observe that the highest powers of T for all components of x appear in the term $i = 1$. Second, the first component for any x is completely determined by the same term mentioned before, and therefore, if the system is proper, B_1 can not be the zero matrix. From both comments we see that things depend primarily on the first row of B.

Also an approximate method to calculate the sets S_T is given for the normal case.

Section 1

Consider a control system of the form

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in E_n$ (real Euclidean space), A, B are $n \times n$ and $n \times r$ real constant matrices respectively, $n \geq 1$, $r \geq 1$, $u \in E_r$ is a real function of t, measurable for $0 \leq t < \infty$, $\| u(t) \| \leq 1$, and $\| u \| = \max_{i=1,\cdots,r} | u_i |$. The class of functions $u(t)$ with the above properties will be denoted by Ω.

For any $u \in \Omega$ the solution of (1) with initial conditions $x = 0$, $t = 0$ is given by

$$x = e^{At} \int_0^t e^{-A\tau} Bu(\tau)\, d\tau.$$

Let us define the set S_T by

$$S_T = E\left(x \in E_n\, ; x = e^{AT} \int_0^T e^{-A\tau} Bu(\tau)\, d\tau, \quad u \in \Omega\right)$$

for $T > 0$. Instead of S_T the set $\mathfrak{A}_T$ will be considered:

$$S_T = e^{AT}\mathfrak{A}_T \tag{2}$$

i.e. $\mathfrak{A}_T = E(y \in E_n\, ; e^{AT}y \in S_T)$.

It is a well-established fact that $\mathfrak{A}_T$ is compact and convex for all $T > 0$ ([1]). Evidently, $\mathfrak{A}_T$ is symmetrical with respect to the origin, i.e. if $y \in \mathfrak{A}_T$, then also $-y \in \mathfrak{A}_T$.

Suppose further that the system (1) is proper, i.e. the vectors $b^{(1)}, \cdots, b^{(r)}, Ab^{(1)}, \cdots, Ab^{(r)}, \cdots, A^{n-1}b^{(1)}, \cdots, A^{n-1}b^{(r)}$, where $b^{(i)}$ represents the ith column of B, generate the whole E_n. Then $\mathcal{C}_T$ contains the origin in its interior for any $T > 0$, (see [1]). An objective of this paper is to present a method to calculate approximately the set $\mathcal{C}_T$. Then S_T can be calculated from (2).

As the set $\mathcal{C}_T$ is convex, to every point x of its frontier there corresponds at least one supporting hyperplane. With this hyperplane the unit normal vector η_x, directed into the halfspace not containing $\mathcal{C}_T$, will be associated. η_x will be called a supporting vector associated with x.

Up to the end of this section it will be supposed that (1) is a normal system, i.e., for each j, $j = 1, \cdots, r$, the vectors $b^{(j)}, Ab^{(j)}, \cdots, A^{n-1}b^{(j)}$ are linearly independent. Then it is a known fact ([1]) that if x is any point of the frontier of $\mathcal{C}_T$ and if η_x is an associated supporting vector, then

$$x = \int_0^T e^{-A\tau}Bu(\tau)\,d\tau \tag{3}$$

where $u(t) = \text{sgn}[\eta_x' e^{-At}B]$ (if $a = (a_1, \cdots, a_n)$, we define $\text{sgn}\,a = (\text{sgn}\,a_1, \cdots, \text{sgn}\,a_n)$, where $\text{sgn}\,a_i = 1$, if $a_i > 0$, $\text{sgn}\,a_i = -1$, if $a_i < 0$, and $\text{sgn}\,a_i = 0$ if $a_i = 0$). Now if η is an arbitrary vector from E_n with unit length and if $u(t, \eta) = \text{sgn}[\eta' e^{-At}B]$, then the point

$$x_\eta = \int_0^T e^{-At}Bu(t, \eta)\,dt \tag{3'}$$

belongs to the frontier of $\mathcal{C}_T$ and η is its supporting vector. Further, the frontier of $\mathcal{C}_T$ does not contain non-degenerated segments; i.e., if $x \neq y$ are two points of $Fr(\mathcal{C}_T)$, then $\eta_x \neq \eta_y$, which is a consequence of (3) and of uniqueness of solutions of (1). The former is true since for normal systems no component of $\eta' e^{-At}B$ is p.p. zero in any time interval, while if the system is only proper there might exist segments in the frontier of $\mathcal{C}_T$, as a component could be p.p. zero; therefore Theorem 1 (to follow) must be understood for normal systems. From this it follows that (3′) defines a mapping of the unit sphere on the frontier of $\mathcal{C}_T$.

Lemma 1. *Let $f(t)$ be a scalar function continuous in $[0, T]$ and with a finite number of zeros there. Suppose that the sequence $f_j(t)$ converges uniformly to $f(t)$ with $j \to \infty$ in $[0, T]$; then $\text{sgn}[f_j(t)]$ converges almost everywhere to $\text{sgn}[f(t)]$ in $[0, T]$.*

Proof. Let $\epsilon > 0$. Suppose that $f(t)$ has n zeros in $[0, T]$. Let every zero be enclosed in an open interval of length ϵ/n. Let $\alpha = \inf|f(t)|$ in the complement C of the union of the mentioned intervals. Suppose j sufficiently large such that $|f_j(t)| > \alpha/2$ in C. Then $\text{sgn}[f_j(t)] = \text{sgn}[f(t)]$ in C, the measure of C being $T - \epsilon$.

From Lemma 1 one obtains that the function $\eta \to x_\eta$ defined by (3′) is continuous and this implies

Lemma 2. *If the set $\{\eta_j\}$ is dense on the frontier of the unit sphere, then the set $\{x_{\eta_j}\}$ is dense on $Fr(\mathcal{C}_T)$.*

THEOREM 1. *Let* $\{\eta_j\}$, $j = 1, 2, \cdots$ *be dense on the frontier of the unit sphere and let* P_s *be the polyhedron generated by the points* $(x_{\eta_1}, \cdots, x_{\eta_s})$, $s = 1, 2, \cdots$; *then* $\overline{\bigcup_{s=1}^{\infty} P_s} = \mathfrak{A}_T$ (*the bar denoting the closure*).

Proof. Suppose $\overline{\bigcup_{s=1}^{\infty} P_s} \subset \mathfrak{A}_T$, $\overline{\bigcup_{s=1}^{\infty} P_s} \neq \mathfrak{A}_T$. Then there exists an interior point p of $\mathfrak{A}_T$ and a neighborhood V of this point which is contained in $\mathfrak{A}_T$, $V \cap \overline{\bigcup_{s=1}^{\infty} P_s} = \emptyset$. Consider the cone generated by V with the vertex in some x_{η_i}. The intersection of the interior of this cone with $Fr(\mathfrak{A}_T)$ necessarily contains a point x_{η_j}, as by Lemma 2 the set $\{x_{\eta_j}\}$ is dense on $Fr(\mathfrak{A}_T)$. Then the polyhedron P_k with $k = \max[i, j]$ contains points of V, which contradicts the hypothesis.

Note 1. Theorem 1 allows to approximate (with an arbitrary degree of precision) the set $\mathfrak{A}_T$ by means of polyhedrons contained in $\mathfrak{A}_T$. Moreover, if P_s is the polyhedron generated by the points $(x_{\eta_1}, \cdots, x_{\eta_s})$, and

$$P_s' = \bigcap_{k=1}^{s} H_k, \; H_k = (x \in E_n \,;\, \eta_k'(x - x_{\eta_k}) \leq 0)$$

(i.e., P_s' is the intersection of all the closed halfspaces containing the origin which are determined by the hyperplane perpendicular to η_k and tangent to $\mathfrak{A}_T$ at the points x_{η_k}), then evidently $P_s \subset \mathfrak{A}_T \subset P_s'$. It is easy to prove an analogous result for proper systems as well.

Section 2

In this section the asymptotic behaviour of the sets S_T for T large will be studied. For this purpose, suppose that the system (1) has been transformed so that A has the Jordan canonical form. Obviously the transformed sytem will be proper again; and to each real root γ_j of A, or to each pair of complex conjugate roots $\alpha_j \pm i\beta_j$, there corresponds an independent subsystem of (1) which is obviously proper again.

Now suppose that there exists a characteristic root of A with real part zero. The corresponding subsystem is of the form

$$\dot{y} = Dy + \tilde{B}u \tag{4}$$

where either

$$D = \begin{pmatrix} 0, & 0, & \cdots, & 0 \\ 1, & 0, & \cdots, & 0 \\ 0, & 1, & \cdots, & 0 \\ \vdots & & & \vdots \\ 0, & \cdots, & 1, & 0 \end{pmatrix} \tag{5}$$

or

$$D = \begin{pmatrix} S_2, & 0_2, & \cdots, & 0_2 \\ I_2, & S_2, & \cdots, & 0_2 \\ \vdots & & & \vdots \\ 0_2 & \cdots, & 0_2, I_2, & S_2 \end{pmatrix} \tag{6}$$

where

$$S_2 = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad 0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In what follows by the *distance* $d_m(\mathfrak{A}_1, \mathfrak{A}_2)$ of two compact non-empty sets $\mathfrak{A}_1 \subset E_k$ and $\mathfrak{A}_2 \subset E_k$ is meant

$$\max\{\sup_{x \in \mathfrak{A}_2} d(x, \mathfrak{A}_1), \sup_{x \in \mathfrak{A}_1} d(x, \mathfrak{A}_2)\},$$

where $d(x, \mathfrak{A}_i)$ means the Euclidean distance between x and $\mathfrak{A}_i$.

Obviously $d_m(\mathfrak{A}_1, \mathfrak{A}_2)$ is zero if and only if $\mathfrak{A}_1 = \mathfrak{A}_2$. Further, if $\mathfrak{A}_i$ are convex, it is easy to show that

$$d_m(Fr(\mathfrak{A}_1), Fr(\mathfrak{A}_2)) = d_m(\mathfrak{A}_1, \mathfrak{A}_2)$$

where $Fr(\mathfrak{A}_i)$ denotes the frontier of $\mathfrak{A}_i$.

THEOREM 2. *Let* $T \geq 1$ *and*

$$S_T = E\left(x; x = \int_0^T e^{D(T-\tau)} Bu(\tau)\, d\tau, u \in \Omega\right).$$

Then there exist convex compact sets $\tilde{\mathfrak{A}}_T$, $\mathfrak{A}_\infty$, *which are symmetrical with respect to the origin and contain a neighborhood of the origin,* $\mathfrak{A}_\infty$ *being independent of* T, $d_m(\tilde{\mathfrak{A}}_T, \mathfrak{A}_\infty) \to 0$ *with* $T \to \infty$, *such that* $x \in Fr(S_T)$ *if and only if* $x = e^{DT} D_T w$, $w \in Fr(\tilde{\mathfrak{A}}_T)$, *where* D_T *is a diagonal matrix whose diagonal elements are natural powers of* T. *Moreover,* $\mathfrak{A}_\infty$ *is independent from* β *and depends only from the first two rows of* B *in the case* (6) *and from the first row of* B *in the case* (5).

COROLLARY. *For every* $x \in Fr(S_T)$ *there exist a* $y \in Fr(\mathfrak{A}_\infty)$ ($y \neq 0$ *as* $\mathfrak{A}_\infty$ *contains the origin in its interior*) *and a* $z(T)$ *such that*

$$x = e^{DT} D_T(y + z(T)),$$

where $\| z(T) \| \leq \rho(T)$, $\rho(T)$ *being independent from* $x \in Fr(S_T)$, $\rho(T) \to 0$ *with* $T \to \infty$.

Proof. Take $w \in Fr(\tilde{\mathfrak{A}}_T)$ which corresponds to x. As $Fr(\mathfrak{A}_\infty)$ is compact there exists a $y \in Fr(\mathfrak{A}_\infty)$ such that $d(w, y) = d(w, Fr(\mathfrak{A}_\infty))$. Now $\| w - y \| = d(w, y) \leq \sup_{w \in \tilde{\mathfrak{A}}_T} d[w, Fr(\mathfrak{A}_\infty)] \leq d_m[Fr(\tilde{\mathfrak{A}}_T), Fr(\mathfrak{A}_\infty)] = d_m(\tilde{\mathfrak{A}}_T, \mathfrak{A}_\infty)$; taking $z(T) = w - y$ and $\rho(T) = d_m(\tilde{\mathfrak{A}}_T, \mathfrak{A}_\infty)$ the corollary is proved.

Theorem 2 will be proved by means of two lemmas.

LEMMA 3. *Let* $K_1 = E(\eta \in E_{2k}; \| \eta \| = 1)$; *let*

$$Q(T, \tau) = \begin{pmatrix} D_2, & 0_2, & 0_2, \cdots, & 0_2 \\ -\tau D_2, & D_2, & 0_2, & 0_2 \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{k-1} \dfrac{\tau^{k-1}}{(k-1)!} D_2, & \dfrac{(-1)^{k-2}}{T} \dfrac{\tau^{k-2}}{(k-2)!} D_2, & \dfrac{(-1)^{k-3}}{T^2} \dfrac{\tau^{k-3}}{(k-3)!} D_2, \cdots, & \dfrac{1}{T^{k-1}} D_2 \end{pmatrix}$$

be a 2k by 2k matrix

$$D_2 = \begin{pmatrix} \cos \beta T\tau & -\sin \beta T\tau \\ \sin \beta T\tau & \cos \beta T\tau \end{pmatrix}, \; T \geq 1, \tau \in [0, 1],$$

β a positive constant, B a $2k \times r$ real constant matrix, $k, r \geq 1$. Let the elements of a matrix C be denoted by $(C)_{ij}$. Then

$$\lim_{T \to \infty} \int_0^t \sum_{j=1}^{r} \left| \sum_{i=1}^{2k} \eta_i (Q(T, \tau)B)_{ij} \right| d\tau = L_\eta$$

exists uniformly with respect to $\eta \in K_1$, where $t \in [0, 1]$, and is independent from β and $(B)_{ij}$, where $i = 3, \cdots, 2k$ and $j = 1, \cdots, r$. Moreover, $L_\eta = \sum_{j=1}^{r} \int_0^t P_j(\tau)\, d\tau$ where

$$P_j(\tau) = \left[\left(\eta_1 - \tau\eta_3 + \cdots + (-1)^{k-1} \frac{\tau^{k-1}}{(k-1)!} \eta_{2k-1} \right)^2 + \left(\eta_2 - \tau\eta_4 + \cdots + (-1)^{k-1} \frac{\tau^{k-1}}{(k-1)!} \eta_{2k} \right)^2 \right]^{1/2} [(B)_{1j}^2 + (B)_{2j}^2]^{1/2}.$$

Proof. Let $B = B_1 + B_2$ and $(B_1)_{ij} = (B)_{ij}$ for $i = 1, 2$; $(B_1)_{ij} = 0$ for $i = 3, \cdots, 2k$; $(B_2)_{ij} = 0$ for $i = 1, 2$; and $(B_2)_{ij} = (B)_{ij}$ for $i = 3, \cdots, 2k$, $j = 1, \cdots, r$. Then $Q(T, \tau)B = Q(T, \tau)B_1 + Q(T, \tau)B_2$; and if the jth column of B_1 is denoted by $B_1^{(j)}$, it holds that

$$Q(T, \tau)B_1^{(j)} = \left(d_j, -\tau d_j, \cdots, (-1)^{k-1} \frac{\tau^{k-1}}{(k-1)!} d_j \right)',$$

where

$$d_j = \begin{pmatrix} (B)_{1j} \cos \beta T\tau - (B)_{2j} \sin \beta T\tau \\ (B)_{1j} \sin \beta T\tau + (B)_{2j} \cos \beta T\tau \end{pmatrix}.$$

It is evident that for $T \geq 1$, $\tau \in [0, 1]$, and $\eta \in K_1$ it holds that

$$\eta' Q(T, \tau)B = \eta' Q(T, \tau)B_1 + \frac{1}{T} r(T, \tau, \eta) \tag{7}$$

where $\| r(T, \tau, \eta) \| \leq c$. Now for each $j = 1, \cdots, r$ one obtains

$$\begin{aligned} \eta'(Q, \tau)B_1^{(j)} &= \left(\eta_1 - \tau\eta_3 + \cdots + (-1)^{k-1} \frac{\tau^{k-1}}{(k-1)!} \eta_{2k-1} \right) [(B)_{1j} \cos \beta T\tau \\ &\quad - (B)_{2j} \sin \beta T\tau] + \left(\eta_2 - \tau\eta_4 + \cdots + (-1)^{k-1} \frac{\tau^{k-1}}{(k-1)!} \eta_{2k} \right) \\ &\quad \cdot [(B)_{1j} \sin \beta T\tau + (B)_{2j} \cos \beta T\tau] \\ &= \left[\left(\eta_1 - \tau\eta_3 + \cdots + (-1)^{k-1} \frac{\tau^{k-1}}{(k-1)!} \eta_{2k-1} \right) (B)_{1j} \right. \\ &\quad \left. + \left(\eta_2 - \tau\eta_4 + \cdots + (-1)^{k-1} \frac{\tau^{k-1}}{(k-1)!} \eta_{2k} \right) (B)_{2j} \right] \cos \beta T\tau \\ &\quad + [(\eta_2 - \cdots)(B)_{1j} - (\eta_1 - \cdots)(B)_{2j}] \sin \beta T\tau = P_{j1}(\tau) \cos \beta T\tau \\ &\quad + P_{j2}(\tau) \sin \beta T\tau. \end{aligned} \tag{8}$$

Let $t \in [0, 1]$, and let m be a positive integer. Then, denoting $t2^{-m} = \Delta$, one obtains

$$\int_0^t | P_{j1}(\tau) \cos \beta T\tau + P_{j2}(\tau) \sin \beta T\tau | \, d\tau = \sum_{k=0}^{2^m-1} \int_{k\Delta}^{(k+1)\Delta} | P_{j1}(\tau_k) \cos \beta T\tau + P_{j2}(\tau_k) \sin \beta T\tau + (P_{j1}(\tau) - P_{j1}(\tau_k)) \cos \beta T\tau + (P_{j2}(\tau) - P_{j2}(\tau_k)) \sin \beta T\tau | \, d\tau,$$

where $\tau_k \in [k\Delta, (k+1)\Delta]$. From this, it follows that

$$\sum_{k=0}^{2^m-1} \left[\int_{k\Delta}^{(k+1)\Delta} | P_{j1}(\tau_k) \cos \beta T\tau + P_{j2}(\tau_k) \sin \beta T\tau | \, d\tau \right] - v \tag{9}$$
$$\leq \int_0^t | P_{j1}(\tau) \cos \beta T\tau + P_{j2}(\tau) \sin \beta T\tau | \, d\tau$$
$$\leq \sum_{k=0}^{2^m-1} \left[\int_{k\Delta}^{(k+1)\Delta} | P_{j1}(\tau_k) \cos \beta T\tau + P_{j2}(\tau_k) \sin \beta T\tau | \, d\tau \right] + v,$$

where $v = \sum_{k=0}^{2^m-1} \int_{k\Delta}^{(k+1)\Delta} | P_{j1}(\tau) - P_{j1}(\tau_k) | + | P_{j2}(\tau) - P_{j2}(\tau_k) | \, d\tau \leq \gamma\Delta^2 2^m = \gamma t^2 2^{-m}$, γ being an upper bound of $dP_{jn}(\tau)/d\tau$ independent from $j = 1, \cdots, r$, $n = 1, 2$, $\eta \in K_1$, and $\tau \in [0, 1]$. Now

$$\int_{k\Delta}^{(k+1)\Delta} | P_{j1}(\tau_k) \cos \beta T\tau + P_{j2}(\tau_k) \sin \beta T\tau | \, d\tau = (P_{j1}^2(\tau_k) + P_{j2}^2(\tau_k))^{1/2} \int_{k\Delta}^{(k+1)\Delta} | \cos(\beta T\tau + \varphi_j) | \, d\tau, \tag{10}$$

where $\varphi_j = -\text{arctg}\, P_{j2}(\tau_k)/P_{j1}(\tau_k) \in [0, 2\pi)$.

Now it will be proved that if a, b, φ are real numbers, $b > a$, and $\varphi \in [0, 2\pi)$, then

$$\lim_{T\to\infty} \int_a^b | \cos(\beta T\tau + \varphi) | \, d\tau = \frac{2}{\pi}(b - a) \tag{11}$$

uniformly with respect to $\varphi \in [0, 2\pi)$.

Let $\{a, \tau_1, \cdots, \tau_p, b\}_T$ be a subdivision of $[a, b]$ (for a fixed T) such that $\beta T\tau_i + \varphi = k_i\pi - (\pi/2)$ for some integer k_i, $k_{i+1} = k_i + 1$, $\beta Ta + \varphi \geq (k_1 - 1)\pi - (\pi/2)$, and $\beta Tb + \varphi \leq (k_p + 1)\pi - (\pi/2)$. Then

$$\int_a^b | \cos(\beta T\tau + \varphi) | \, d\tau = \sum_{i=1}^{p-1} \int_{\tau_i}^{\tau_{i+1}} | \cos(\beta T\tau + \varphi) | \, d\tau + \int_a^{\tau_1} + \int_{\tau_p}^b = (p - 1)\frac{2}{\beta T} + \int_a^{\tau_1} + \int_{\tau_p}^b.$$

It is evident that $\tau_{i+1} - \tau_i = \pi/\beta T$ for $i = 1, \cdots, p - 1$, $\tau_1 - a \leq \pi/\beta T$, $b - \tau_p \leq \pi/\beta T$, and $b - a - (2\pi/\beta T) \leq (p - 1)(\pi/\beta T) \leq b - a$; hence the result follows immediately.

From (10) and (11) it follows that

$$(12)\quad \int_{k\Delta}^{(k+1)\Delta} | P_{j1}(\tau_k) \cos \beta T\tau + P_{j2}(\tau_k) \sin \beta T\tau | \, d\tau \rightarrow (P_{j1}^2(\tau_k) + P_{j2}^2(\tau_k))^{1/2} \frac{2\Delta}{\pi}$$

with $T \rightarrow \infty$ uniformly with respect to $\eta \in K_1$.

By (9) and (12) one has for every $T \geq 1$, $m = 1, 2, \cdots$

$$(13)\quad \begin{aligned} \sum_{k=0}^{2m-1} \left\{ P_j(\tau_k) \frac{2}{\pi} t2^{-m} - \psi(T, m)t2^{-m} \right\} - v &\leq \int_0^t | P_{j1}(\tau) \cos \beta T\tau + P_{j2}(\tau) \sin \beta T\tau | \, d\tau \\ &\leq \sum_{k=0}^{2m-1} \left\{ P_j(\tau_k) \frac{2}{\pi} t2^{-m} + \psi(T, m)t2^{-m} \right\} + v, \end{aligned}$$

where

$$P_j(\tau) = (P_{j1}^2(\tau) + P_{j2}^2(\tau))^{1/2} = \left\{ \left[\left(\eta_1 - \tau\eta_3 + \cdots + (-1)^{k-1} \frac{\tau^{k-1}}{(k-1)!} \eta_{2k-1} \right)^2 + \left(\eta_2 - \tau\eta_4 + \cdots + (-1)^{k-1} \frac{\tau^{k-1}}{(k-1)!} \eta_{2k} \right)^2 \right] [(B)_{1j}^2 + (B)_{2j}^2] \right\}^{1/2},$$

$\psi(T, m) \rightarrow 0$ with $T \rightarrow \infty$ for $m = 1, 2, \cdots$, $\psi(T, m)$ being positive and independent from $\eta \in K_1$, $t \in [0, 1]$.

Clearly (13) implies that

$$(14)\quad \begin{aligned} \frac{2}{\pi} \sum_{k=0}^{2m-1} P_j(\tau_k) - v &\leq \liminf_{T\to\infty} \int_0^t | P_{j1}(\tau) \cos \beta T\tau + P_{j2}(\tau) \sin \beta T\tau | \, d\tau \\ &\leq \limsup_{T\to\infty} \int_0^t | P_{j1}(\tau) \cos \beta T\tau + P_{j2}(\tau) \sin \beta T\tau | \, d\tau \\ &\leq \frac{2}{\pi} \sum_{k=0}^{2m-1} P_j(\tau_k)\Delta + v \end{aligned}$$

for $m = 1, 2, \cdots$, $t \in [0, 1]$. Hence

$$(15)\quad \int_0^t | P_{j1}(\tau) \cos \beta T\tau + P_{j2}(\tau) \sin \beta T\tau | \, d\tau \rightarrow \frac{2}{\pi} \int_0^t P_j(\tau) \, d\tau$$

with $T \rightarrow \infty$. Moreover, from (13) and (14) it follows that

$$\left| \frac{2}{\pi} \int_0^t P_j(\tau) \, d\tau - \int_0^t | P_{j1}(\tau) \cos \beta T\tau + P_{j2}(\tau) \sin \beta T\tau | \, d\tau \right| \leq 2v + \psi(T, m).$$

By (9), $v \leq \gamma^{2-m}$. Now given an $\epsilon > 0$ choose $m = m_1$ such that $2\gamma 2^{-m} \leq \epsilon/2$ and T_1 such that $\psi(T, {}_{m1}) < \epsilon/2$ for all $T \geq T_1$. Neither m_1 nor T_1 depends on

$\eta \in K_1$, $t \in [0, 1]$, which proves the uniform convergence in (15). Since the constant c in (7) is independent from $T \geq 1$, $\tau \in [0, 1]$, and $\eta \in K_1$,

$$\int_0^t \sum_{j=1}^{r} \left| \sum_{i=1}^{2k} \eta_i'(Q(T,\tau)B)_{ij} \right| d\tau \to \frac{2}{\pi} \sum_{j=1}^{r} \int_0^t P_j(\tau)\, d\tau,$$

with $T \to \infty$ uniformly with respect to $\eta \in K_1$, and $t \in [0, 1]$.

LEMMA 4. *Let the hypothesis of Lemma 3 be fulfilled. For $T \geq 1$ let $\tilde{\mathfrak{A}}_T = E(x \in E_{2k}\,;\, x = \int_0^1 Q(T, \tau)Bu(\tau)\, d\tau,\ u \in \Omega)$. Then $\tilde{\mathfrak{A}}_T$ is a compact convex set, and there exists a compact convex set $\mathfrak{A}_\infty \subset E_{2k}$, which is symmetric with respect to the origin, such that $d_m(\tilde{\mathfrak{A}}_T, \mathfrak{A}_\infty) \to 0$ with $T \to \infty$. Moreover, $\mathfrak{A}_\infty$ does not depend on β, T and depends on the first two rows of B only, and if at least one element of the first pair of rows of B is different from zero, then $\mathfrak{A}_\infty$ contains the origin in its interior.*

Proof. Since $\tilde{\mathfrak{A}}_T$ is obtained from $\mathfrak{A}_T$ in (2) by means of a regular linear transformation with the transformation matrix $\operatorname{diag}(T, T, T^2, T^2, \cdots, T^k, T^k)$, $\tilde{\mathfrak{A}}_T$ is compact, convex, and contains the origin in its interior. Now let $\eta \in K_1$. Then

$$y_\eta^T = \int_0^1 Q(T, \tau)B \operatorname{sgn}(\eta' Q(T, \tau)B)\, d\tau$$

is evidently a point of contact of the hyperplane tangent to $\tilde{\mathfrak{A}}_T$ and perpendicular to η. By Lemma 3, given $\epsilon > 0$ there is a T_ϵ such that for $T > T_\epsilon$

$$| \eta' y_\eta^T - L_\eta | < \epsilon$$

for all $\eta \in K_1$.

Let $\mathfrak{A}_\epsilon^+ = \bigcap_{\eta \in K_1} H_{L_\eta + \epsilon}$, $\mathfrak{A}_\epsilon^- = \bigcap_{\eta \in K_1} H_{L_\eta - \epsilon}$, $\mathfrak{A}_\infty = \bigcap_{\eta \in K_1} H_{L_\eta}$ where $H_{L_\eta + \sigma} = E(x \in E_{2k}\,;\, \eta' x \leq L_\eta + \sigma)$ for $\sigma \in E_1$. It is easy to see that $\mathfrak{A}_\epsilon^+$, $\mathfrak{A}_\epsilon^-$, $\mathfrak{A}_\infty$ are compact, convex and symmetrical with respect to the origin, $\mathfrak{A}_\epsilon^- \subseteq \tilde{\mathfrak{A}}_T \subseteq \mathfrak{A}_\epsilon^+$ for $T \geq T_\epsilon$; moreover, $\mathfrak{A}_\epsilon^- \subseteq \mathfrak{A}_\infty \subseteq \mathfrak{A}_\epsilon^+$. But $d_m(\mathfrak{A}_\epsilon^-, \mathfrak{A}_\epsilon^+) \to 0$ with $T \to \infty$ and $d_m(\tilde{\mathfrak{A}}_T, \mathfrak{A}_\infty) \leq d_m(\mathfrak{A}_\epsilon^+, \mathfrak{A}_\epsilon^-)$. Now, if $\sum_{j=1}^{r} (B)_{1j}^2 + (B)_{2j}^2 \neq 0$, by Lemma 3 $L_\eta > 0$ for every $\eta \in K_1$; as K_1 is compact, $L_\eta > \alpha > 0$ for every $\eta \in K_1$ and $\mathfrak{A}_\infty$ contains the origin in its interior. Lemma 4 is proved.

Proof of Theorem 2. Let D be of type (5). Then for $x \in S_T$ one has: $x = e^{DT} \operatorname{diag}(T, T^2, \cdots, T^k) \int_0^1 \tilde{Q}(T, \tau)\tilde{B}u(\tau)\, d\tau$, where

$$\tilde{Q}(T, \tau) = \begin{Bmatrix} 1, & 0, & & & \cdots, & 0 \\ -\tau, & \dfrac{1}{T}, & 0, & & \cdots, & 0 \\ \dfrac{\tau^2}{2!}, & -\dfrac{\tau}{T}, & \dfrac{1}{T^2}, & 0, & \cdots, & 0 \\ \vdots & & & & & \vdots \\ \dfrac{(-1)^{k-1}}{(k-1)!}\tau^{k-1}, & & & & \cdots, & \dfrac{1}{T^{k-1}} \end{Bmatrix}$$

Let $\tilde{B} = \{\tilde{b}_{ij}\} = \tilde{B}_1 + \tilde{B}_2$ where $\tilde{B}_1$ has the same first row as $\tilde{B}$ and the remaining rows are zero.

Evidently, $x = e^{DT} \operatorname{diag}(T, \cdots, T^k)[\int_0^1 \tilde{Q}_1(\tau)u(\tau)\, d\tau + (1/T)q_1(T, u)]$, where $\tilde{Q}_1(\tau)$ is bounded, depends on $\tilde{B}_1$ but not on $\tilde{B}_2$ and T; also $\| q_1(T, u) \| \leq \lambda$, λ being independent of $T \geq 1$ and $u \in \Omega$. Let $\tilde{\mathfrak{A}}_T = E(w \in E_k ; w = \int_0^1 \tilde{Q}(T, \tau)\tilde{B}u(\tau)\, d\tau,\ u \in \Omega)$ and $\mathfrak{A}_\infty = E(w \in E_k ; w = \int_0^1 \tilde{Q}_1(\tau)u(\tau)\, d\tau$, $u \in \Omega)$. $\tilde{\mathfrak{A}}_T$ is obtained from $\mathfrak{A}_T$ by means of a regular linear transformation and, therefore, is compact, convex, and symmetrical with respect to the origin and contains a neighborhood of the origin for $T \geq 1$. To prove that $\mathfrak{A}_\infty$ has the desired properties it is sufficient, since

$$\mathfrak{A}_\infty = E\left(w \in E_k ; w = \int_0^1 e^{-D\tau}\tilde{B}_1 u(\tau)\, d\tau, \quad u \in \Omega\right),$$

to prove that the system

$$\dot{x} = Dx + \tilde{B}_1 u \tag{16}$$

is proper. First of all, observe that if $\sum_{j=1}^{r} | \tilde{b}_{1j} | = 0$ then the first component x_1 of the vector

$$\int_0^t e^{-D\tau}\tilde{B}u(\tau)\, d\tau$$

is zero, which implies that the system (4) is not proper. Thus $\tilde{b}_{1j} \neq 0$ at least for one $j = 1, \cdots, r$, say for j_1. Then $D^n(\tilde{b}_{1_{j_1}}, 0, \cdots, 0)' = (0, 0, \cdots, \tilde{b}_{1_{j_1}}, 0, \cdots, 0)'$ for $n = 0, 1, \cdots, k - 1$, the non-zero element in the right side being in the $(n + 1)$th place. Consequently, $(\tilde{b}_{1_{j_1}}, 0, \cdots, 0)'$, $(0, \tilde{b}_{1_{j_1}}, 0, \cdots, 0)'$, $\cdots$, $(0, 0, \cdots, 0, \tilde{b}_{1_{j_1}})'$ form a complete system of linearly independent vectors and the system (16) is proper.

From the definition of $\mathfrak{A}_\infty$ and $\tilde{\mathfrak{A}}_T$ it follows immediately that $d_m(\tilde{\mathfrak{A}}_T, \mathfrak{A}_\infty) \leq (1/T)\lambda$ which proves Theorem 2 for the case that D is of type (5).

Now let D be of type (6). Evidently, for $T \geq 1$ the matrix $e^{DT} \operatorname{diag}(T, T, T^2, T^2, \cdots, T^k, T^k)$ determines a one-to-one correspondence between the points of $Fr(\tilde{\mathfrak{A}}_T)$ and $Fr(S_T)$. Further, $\sum_{j=1}^{r} (\tilde{B})_{1j}^2 + (\tilde{B})_{2j}^2 = 0$ implies similarly as in the case (5) that the system (4) is not proper. Thus $\sum_{j=1}^{r} (\tilde{B})_{1j}^2 + (\tilde{B})_{2j}^2 \neq 0$ and, by Lemma 4, $\mathfrak{A}_\infty$ contains a neighborhood of the origin.

Note 2. It can be easily seen that the sets $\tilde{\mathfrak{A}}_T$, $\mathfrak{A}_\infty$ can be calculated approximately in the same way as the set $\mathfrak{A}_T$ in Note 1. It is also obvious that Theorem 2 may be generalized for the case that the matrix A consist of several Jordan blocks (5) and (6).

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Reference

[1] LaSalle, J. P. *The time optimal control problem*, Contributions to the Theory of Nonlinear Oscillations, Vol. 5, Princeton, 1960.