ABSOLUTE CESARO SUMMABILITY OF A SERIES ASSOCIATED WITH THE CONJUGATE SERIES OF A FOURIER SERIES

By S. M. Mazhar

1.1. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Let its Fourier series be

$$\tfrac{1}{2}a_0 + \textstyle\sum_1^\infty (a_n \operatorname{Cos} nt + b_n \operatorname{Sin} nt) = \sum_0^\infty A_n(t);$$

then the series conjugate to it is

$$\textstyle\sum_1^\infty (b_n \operatorname{Cos} nt - a_n \operatorname{Sin} nt) = \sum_1^\infty B_n(t).$$

We write

$$\varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\};$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\};$$

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}\psi(u)\,du, \qquad \alpha > 0;$$

$$\Psi_0(t) = \psi(t).$$

We employ $\Phi_\alpha(t)$ with similar meaning.

1.2. Matsumoto [2] in 1956 obtained the following theorem concerning the absolute Cesàro summability of a series associated with a Fourier series.

Theorem A. *If*

(i) $\Phi_\beta(+0) = 0$,

(ii) $\int_0^\pi t^{-\gamma} \,|\, d\Phi_\beta(t) \,|\, < \infty$,

then the series $\sum n^{\gamma-\beta} A_n(t)$, *at the point* $t = x$, *is summable* $|\, C, \alpha \,|$, *where* $1 > \alpha > \gamma \geqslant \beta \geqslant 0$.

The object of this note is to prove the corresponding theorem for a series associated with the conjugate series of a Fourier series.

2.1. In what follows we shall prove the following theorem:

Theorem. *If*

(i) $\Psi_\beta(+0) = 0$,

(ii) $\int_0^\pi t^{-\gamma} \,|\, d\Psi_\beta(t) \,|\, < \infty$, *then the series* $\sum n^{\gamma-\beta}B_n(t)$, *at the point* $t = x$, *is summable* $|C, \alpha|$, *where* $1 > \alpha > \gamma \geqslant \beta > 0$ *and also* $1 > \alpha > \gamma > \beta \geqslant 0$.*

It may be observed that, for the special case $\gamma = \beta$, our theorem includes the following theorem of Bosanquet and Hyslop [1].

THEOREM B. *If* $0 < \beta < 1$, *and*

(i) $\Psi_\beta(+0) = 0$,

(ii) $\int_0^\pi t^{-\beta} \, | d\Psi_\beta(t)| < \infty$,

then the conjugate series $\sum B_n(t)$, *at the point* $t = x$, *is summable* $|C, \alpha|$, *for every* $\alpha > \beta$.

On the other hand, if we consider the case $\beta = 0$, then it reduces to the following theorem of Mohanty [3].

THEOREM C. *If* $0 < \gamma < 1$, *and*

(i) $\psi(+0) = 0$,

(ii) $\int_0^\pi t^{-\gamma} \, | d\psi(t)| < \infty$,

then the series $\sum n^\gamma B_n(t)$, *at the point* $t = x$, *is summable* $|C, \alpha|$ *for* $\alpha > \gamma$.

2.2. The following lemmas will be required for the proof of our theorem.

LEMMA 1. *Let*

$$S_k(n, t) = \sum_{\nu=0}^{k} A_{n-\nu}^{\alpha-1} \operatorname{Cos} \nu t, \qquad k \leqslant n, \; 0 < \alpha < 1.$$

Then we have

$$S_k(n, t) = \begin{cases} O\{k(n-k)^{\alpha-1}\} \\ O\{t^{-1}(n-k)^{\alpha-1}\} \end{cases} \qquad n > k,$$

and

$$S_n(n, t) = \begin{cases} O(n^\alpha) \\ O(t^{-\alpha}). \end{cases}$$

Proof.

$$\begin{aligned} S_k(n, t) &= \sum_{\nu=0}^{k} A_{n-\nu}^{\alpha-1} \operatorname{Cos} \nu t \\ &\leqslant \sum_{\nu=0}^{k} A_{n-\nu}^{\alpha-1} \\ &= O\{k(n-k)^{\alpha-1}\}. \end{aligned}$$

Also

$$\begin{aligned} S_k(n, t) &\leqslant 2A_{n-k}^{\alpha-1} \operatorname{Max}_{N,N'} \left| \sum_{N}^{N'} \operatorname{Cos} \nu t \right| \\ &= O\{t^{-1}(n-k)^{\alpha-1}\}. \end{aligned}$$

* It should be noted that this theorem is not true for the case $\gamma = \beta = 0$; see Bosanquet and Hyslop [1].

The second part of the lemma is due to Obrechkoff [4].

LEMMA 2. *Let*

$$S^{\lambda}{}_k(n, t) = \left(\frac{d}{dt}\right)^{\lambda} S_k(n, t).$$

Then we have

$$S_k{}^{\lambda}(n, t) = \begin{cases} O\{k^{\lambda+1}(n-k)^{\alpha-1}\} \\ O\{t^{-1}k^{\lambda}(n-k)^{\alpha-1}\} \end{cases} \qquad k < n,$$

and

$$S_n{}^{\lambda}(n, t) = \begin{cases} O(n^{\alpha+\lambda}) \\ O(n^{\lambda}t^{-\alpha}). \end{cases}$$

The proof of the first part of Lemma 2 is similar to that of Lemma 1. For the proof of the second part, see Obrechkoff [4].

LEMMA 3. *Let*

$$H^{\alpha}(n, t) = \frac{1}{A_n{}^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}{}^{\alpha-1} \nu^{\delta} \cos \nu t$$

where $\delta = \gamma - \beta$, for $\gamma > \beta \geqslant 0$, *then*

$$H^{\alpha}(n, t) = \begin{cases} O(n^{\delta}) \\ O\{t^{-1}n^{\delta-1} + t^{-\alpha}n^{\delta-\alpha}\}, \end{cases}$$

and

$$\left(\frac{d}{dt}\right)^{\lambda} H^{\alpha}(n,t) = \begin{cases} O(n^{\delta+\lambda}) \\ O\{t^{-1}n^{\lambda+\delta-1} + t^{-\alpha}n^{\lambda+\delta-\alpha}\}. \end{cases}$$

LEMMA 4. *Let*

$$J(n, u) = \int_u^{\pi} (t-u)^{-\beta} \frac{d}{dt} H^{\alpha}(n, t)\, dt, \qquad \delta = \gamma - \beta, \qquad \gamma > \beta > 0.$$

Then

$$J(n, u) = \begin{cases} O(n^{\delta+\beta}) \\ O\{u^{-1}n^{\delta+\beta-1} + u^{-\alpha}n^{\delta+\beta-\alpha}\}. \end{cases}$$

These two lemmas can be easily obtained by using the method of Matsumoto [2].

3.1. Proof of the theorem. It is sufficient to prove that

$$\textstyle\sum_1^{\infty} |\, \xi_n{}^{\alpha} \,|/n < \infty,$$

where

$$\begin{aligned}
\xi_n^{\alpha} &= \frac{1}{A_n^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} \nu\nu^{\gamma-\beta} B_\gamma(x) \\
&= \frac{1}{A_n^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} \nu\nu^{\gamma-\beta} \cdot \frac{2}{\pi} \int_0^{\pi} \psi(t) \text{ Sin } \nu t \, dt \\
&= \frac{2}{\pi} \int_0^{\pi} \psi(t) \frac{1}{A_n^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} \nu\nu^{\gamma-\beta} \sin \nu t \, dt \\
&= -\frac{2}{\pi} \int_0^{\pi} \psi(t) \frac{d}{dt} \left\{ \frac{1}{A_n^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} \nu^{\gamma-\beta} \text{ Cos } \nu t \right\} dt \\
&= -\frac{2}{\pi} \int_0^{\pi} \psi(t) \frac{d}{dt} H^{\alpha}(n, t) \, dt \\
&= -\frac{2}{\pi} \int_0^{\pi} \frac{d}{dt} H^{\alpha}(n, t) \left\{ \frac{1}{\Gamma(1-\beta)} \int_0^{t} (t-u)^{-\beta} \, d\Psi_\beta(u) \right\} dt,
\end{aligned}$$

since $\Psi_\beta(+0) = 0$,

$$\begin{aligned}
&= -\frac{2}{\pi} \int_0^{\pi} d\Psi_\beta(u) \frac{1}{\Gamma(1-\beta)} \int_u^{\pi} (t-u)^{-\beta} \frac{d}{dt} H^{\alpha}(n, t) \, dt \\
&= -\frac{2}{\pi\Gamma(1-\beta)} \int_0^{\pi} d\Psi_\beta(u) J(n, u).
\end{aligned}$$

Therefore

$$\sum_1^{\infty} | \xi_n^{\alpha} |/n = O\left\{ \int_0^{\pi} \sum_1^{\infty} \frac{| J(n, u) |}{n} | d\Psi_\beta(u) | \right\}.$$

Now

$$\sum\nolimits_1^{\infty} | J(n, u) |/n = \sum\nolimits_{n \leqq u^{-1}} + \sum\nolimits_{n > u^{-1}} = M_1 + M_2, \text{ say.}$$

From Lemma 4, we have

$$\begin{aligned}
M_1 &= \sum\nolimits_{n \leqq u^{-1}} O(n^{\delta+\beta-1}) \\
&= O\{ \int_0^{u^{-1}} y^{\delta+\beta-1} dy \} = O(u^{-\gamma}).
\end{aligned}$$

Also

$$\begin{aligned}
M_2 &= \sum\nolimits_{n > u^{-1}} O\{ n^{\delta+\beta-1} (nu)^{-\alpha} \} \\
&= O\{ u^{-\alpha} \int_{u^{-1}}^{\infty} y^{\delta+\beta-1-\alpha} dy \} \\
&= O(u^{-\gamma}).
\end{aligned}$$

Therefore

$$\sum\nolimits_1^{\infty} |\xi_n^{\alpha}|/n = O\{ \int_0^{\pi} u^{-\gamma} | d\Psi_\beta(u) | \} < \infty,$$

by hypothesis.

This completes the proof of the theorem.

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