LYAPUNOV STABILITY AND PERIODIC SOLUTIONS

BY JANE CRONIN*

A well-known method for establishing the existence of periodic solutions of nonlinear ordinary differential equations is to apply the Poincaré-Bendixson Theorem to autonomous equations or the Brouwer Fixed Point Theorem to nonautonomous equations. (See Lefschetz [5, Ch. XI] for a detailed discussion and references.) The lengthiest part of the method is the construction of a Jordan curve J such that all paths ultimately stay inside J. Here an analogous approach is used to obtain periodic solutions for two-dimensional systems. The step corresponding to the construction of J referred to above is carried out by applying extensions of the Lyapunov stability theory due to Malkin [6, 7, 8] to the study of the stability of the Bendixson point at infinity. This approach is related to work of Gomory [3], who used the Poincaré line at infinity. The theorems obtained are related to, but do not overlap, the theorems described by Lefschetz [5, Ch. XI]. A result similar to those obtained here, although with somewhat different hypotheses, is derived by Halanay [4].

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I. Stability of the point at infinity and periodic solutions

Consider the autonomous system

$$\text{(E)} \qquad \begin{aligned} \dot{x}_1 &= X_1(x_1, x_2) \\ \dot{x}_2 &= X_2(x_1, x_2) \end{aligned}$$

where X_1 and X_2 are of class C^1 in x_1 and x_2 at all points in the (x_1, x_2)-plane. Let C_R denote the circle

$$[(x_1, x_2)/x_1^2 + x_2^2 = R^2]$$

and $[C_R]$ denote its interior.

DEFINITION 1. The point at infinity is *strongly stable relative to* (E) if given a positive number R, there is a Jordan curve J whose interior contains C_R and all the critical points of (E) and is such that each path of (E) which intersects J crosses J going outward. (Direction on a path means direction of increasing t.)

LEMMA 1. *If* $p_0 = (x_1^{(0)}, x_2^{(0)})$ *is an asymptotically stable critical point of* (E) *and* $\epsilon > 0$, *there is a Jordan curve* J_1, *differentiable and of finite length, such that* p_0 *is in the interior of* J_1, *the points of* J_1 *are all in the neighborhood* $N_\epsilon(p_0)$, *and any path of* (E) *which intersects* J_1 *crosses* J_1 *going inward.*

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Proof: For simplicity assume $p_0 = \bar{0} = (0, 0)$. By Massera's Theorem (Massera [9] or Malkin [8, p. 312 of the AEC translation]) there is a neighborhood $N_{\epsilon_1}(\bar{0})$ and a function $V(x_1, x_2)$ with domain $N_{\epsilon_1}(\bar{0})$ such that $V(x_1, x_2)$ is positive definite and of class C^1 in x_1 and x_2 in $N_{\epsilon_1}(\bar{0})$ (see Malkin [8, equation (73.8), p. 314]) and the derivative

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

is negative definite in $N_{\epsilon_1}(\bar{0})$. Let $R = \min(\epsilon, \epsilon_1/2)$; let $2r$ be a lower bound for $V(x_1, x_2)$ on C_R; and let

$$F = [(x_1, x_2) \in N_R(\bar{0})/V(x_1, x_2) = r].$$

Since $V(0, 0) = 0$ and $V(x_1, x_2) \geq 2r$ on C_R, there is at least one point of F on each line segment from $\bar{0}$ to C_R. Also F is a closed set and the distance from F to C_R is positive. Let $(x_{10}, x_{20}) \in F$. Since

$$\frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 < 0$$

at (x_{10}, x_{20}), then either $\partial V/\partial x_1$ or $\partial V/\partial x_2$ is nonzero at (x_{10}, x_{20}). Suppose $\partial V/\partial x_1 \neq 0$ at (x_{10}, x_{20}). Then by the Implicit Function Theorem, the equation

$$(Vx_1, x_2) - r = 0$$

has a unique solution $x_1 = f(x_2)$ in some neighborhood N of (x_{10}, x_{20}). That is, the set $N \cap F$ is described by a continuously differentiable curve with no endpoints. Since this kind of argument can be applied to any point in F, the set F is described by a set of curves of finite length. Since each of these curves has no endpoints or multiple points, set F is described by a set of pairwise disjoint Jordan curves. Let J_1 be such a Jordan curve, and suppose that J_1 does not contain $\bar{0}$ in its interior. Since $V(x_1, x_2) = r$ on J_1, then, by the negative definiteness of dV/dt, function V has a minimum less than r or a maximum greater than r in the interior of J_1, and this minimum or maximum is different from zero because $\bar{0}$ is not in the interior of J_1. As this contradicts the negative definiteness of dV/dt, $\bar{0}$ must be interior to J_1. Curve J_1 is the desired curve because, by the negative definiteness of dV/dt, any path of (E) which intersects J_1 crosses J_1 inward.

(Although it is not needed for the proof of the theorem, a similar argument shows that the entire set F is described by curve J_1.)

THEOREM 1. *If the point at infinity is strongly stable relative to* (*E*) *and if* (*E*) *has exactly one critical point* p_0 *and* p_0 *is asymptotically stable, then* (*E*) *has at least one periodic solution.*

Proof: For simplicity let $p_0 = \bar{0}$. Let R be a positive constant. By Definition 1 and Lemma 1, there exist Jordan curves J and J_1 such that J contains the circle C_R and J_1 is contained in the neighborhood $N_{R/2}(\bar{0})$. The region Ω bounded by

J_1 and J contains no critical point of (E); and, since the paths of (E) cross J_1 inward and J outward, region Ω contains a negative half-path. Hence, by the Poincaré-Bendixson Theorem (Lefschetz [5, p. 232, Theorem (9.3)]), region Ω contains a closed path (periodic solution).

Now let the functions $G_1(x_1, x_2, t)$ and $G_2(x_1, x_2, t)$ be of class C^1 in x_1, x_2, and t at all points (x_1, x_2, t), be bounded for all (x_1, x_2, t), and have period T in t. Assume also that at least one of the two functions G_1 and G_2 is explicitly a function of t.

THEOREM 2. *If the point at infinity is strongly stable relative to* (*E*) *and the curve* J *of Definition* 1 *is differentiable and of finite length and if* $|\eta|$ *is sufficiently small, then the system*

$$(\mathrm{E} - \eta) \qquad \begin{aligned} \dot{x}_1 &= X_1(x_1, x_2) + \eta G_1(x_1, x_2, t) \\ \dot{x}_2 &= X_2(x_1, x_2) + \eta G_1(x_1, x_2, t) \end{aligned}$$

has at least one solution of period T.

Proof: There is a Jordan curve J with the additional property that, for sufficiently small $|\eta|$, there are no critical points of $(\mathrm{E} - \eta)$ on J. Let $\phi(p, \eta)$ be the angle from the outward normal to J at point p to the segment of a solution curve of $(\mathrm{E} - \eta)$ which starts at p and proceeds in the direction of increasing t. From the continuity of $\phi(p, 0)$ on J there is a positive number a such that for all $p \in J$

$$0 \leq \phi(p, 0) \leq \frac{\pi}{2} - a.$$

Since, for $\eta \neq 0$, system $(\mathrm{E} - \eta)$ is nonautonomous, then, if $\eta \neq 0$, more than one solution curve of $(\mathrm{E} - \eta)$ may pass through the point p. That is, if $\eta \neq 0$, function $\phi(p, \eta)$ is not generally single-valued. However, it is easy to show that there is a positive number b such that for all $p \in J$ and all $|\eta| < b$

$$|\phi(p, \eta) - \phi(p, 0)| < \frac{a}{2}.$$

Hence the curves described by solutions of $(\mathrm{E} - \eta)$ must cross J going outward.

Now let η be fixed and such that $|\eta| < b$, and let $u(t, p_0, t_0)$ be the solution of $(\mathrm{E} - \eta)$ through p_0 at time t_0. Let p be any point interior to or on J. The mapping M defined by

$$M: p \to u(-T, p, 0)$$

is defined for all such p because $u(t, p, 0)$ stays inside J for all $t \leq 0$. Thus M is a continuous mapping of σ, the 2-cell bounded by J, into itself. Hence by the Brouwer Fixed Point Theorem, there is at least one point $p_1 \in \sigma$ such that

$$p_1 = u(-T, p_1, 0)$$

or

$$u(0, p_1, 0) = u(-T, p_1, 0).$$

Since $G_1(x_1, x_2, t)$ and $G_2(x_1, x_2, t)$ have period T in t, solution $u(t, p_1, 0)$ has period T.

Note: Theorem 1 is obtained only for the 2-dimensional case because the Poincaré-Bendixson Theorem is valid only for the 2-dimensional case. However the definition of strong stability of the point at infinity and Theorem 2 can be formulated for the n-dimensional case. Only the 2-dimensional case is described because this is the only case for which there exist practical criteria for determining if the point at infinity is strongly stable (see Part II).

II. Criteria for strong stability of the point at infinity

Assume that (E) has the form

$$(\mathrm{E_p}) \qquad \begin{aligned} \dot{x}_1 &= \textstyle\sum_{j=0}^{m} X_{1j}(x_1, x_2) \\ \dot{x}_2 &= \textstyle\sum_{j=0}^{m} X_{2j}(x_1, x_2), \end{aligned}$$

where X_{ij} is a form of degree j for $i = 1, 2$, and that the set of critical points of $(\mathrm{E_p})$ is bounded. Under the inversion transformation

$$I: (x_1, x_2) \rightarrow (y_1, y_2)$$

defined by

$$y_i = \frac{x_i}{x_1^2 + x_2^2} \; (i = 1, 2),$$

system $(\mathrm{E_p})$ becomes

$$(\mathrm{I} - \mathrm{E_p}) \qquad \dot{y}_i = \frac{Y_i(y_1, y_2)}{(y_1^2 + y_2^2)^m} \; (i = 1, 2)$$

where the Y_i are polynomials. Except at the origin and the other critical points of the system

$$(\mathrm{S}) \qquad \dot{y}_i = Y_i(y_1, y_2)(i = 1, 2),$$

the paths of $(\mathrm{I} - \mathrm{E_p})$ and (S) are defined at each point of the (y_1, y_2)-plane and coincide.

LEMMA 2. *If the origin $\bar{0}$ in the (y_1, y_2)-plane is an asymptotically stable critical point of (S) and if the set of critical points of $(\mathrm{E_p})$ is bounded, then the point at infinity is strongly stable relative to $(\mathrm{E_p})$ and the curve J of definition 1 is differentiable and of finite length.*

Proof: Let R be a positive number greater than one such that the set of critical points of $(\mathrm{E_p})$ is contained in the circle C_R. Let $\epsilon < 1/R$. Since $\bar{0}$ is asymptotically stable, then, by Lemma 1, there is a Jordan curve J_1 contained in $N_\epsilon(\bar{0})$

such that $\bar{0}$ is in the interior of J_1 and any path of (S) which intersects J_1 crosses J_1 inward. Let $I^{-1}(J_1)$ be the image in the (x_1, x_2)-plane of curve J_1 under the inversion transformation $I^{-1} = I$. Curve $I^{-1}(J_1)$ is a Jordan curve J, differentiable and of finite length, satisfying the conditions in Definition 1.

Thus the problem of determining if the point at infinity is strongly stable relative to (E_p) is reduced to the study of the stability of the critical point $\bar{0}$ of system (S). Since all the terms on the right in (S) are of order higher than one, this study cannot be made by classical means. But we show how results of Malkin [7, 8] can be combined with Theorem 2 to obtain the existence of periodic solutions for nonautonomous systems. A similar application of Theorem 1 to autonomous systems can be made.

First let $\mathfrak{X}_i(y_i, y_2)$ be the homogenous form of lowest order terms in the ith equation in system (S) for $i = 1, 2$. Define the forms:

$$\mathcal{P}(y_1, y_2) = y_1 \mathfrak{X}_1(y_1, y_2) + y_2 \mathfrak{X}_2(y_1, y_2),$$

$$\mathcal{G}(y_1, y_2) = y_2 \mathfrak{X}_1(y_1, y_2) - y_1 \mathfrak{X}_1(y_1, y_2).$$

Malkin's stability conditions (see especially [8, pp. 417–29]) are given in terms of these forms. Combining Malkin's results with Theorem 2 and Lemma 2, we obtain:

THEOREM 3. *Suppose the system*

$$(E_p - \eta) \qquad \begin{aligned} \dot{x}_1 &= X_1(x_1, x_2) + \eta G_1(x_1, x_2, t) \\ \dot{x}_2 &= X_2(x_1, x_2) + \eta G_2(x_1, x_2, t) \end{aligned}$$

satisfies the following conditions:

a) $X_i = \sum_{j=0}^{m} X_{ij}(x_1, x_2)$, *where the* X_{ij} *are forms of degree* j;

b) G_1 *and* G_2 *are of class* C^1 *in* x_1, x_2, *and* t *at every point in* (x_1, x_2, t)-*space and are bounded in* (x_1, x_2, t)-*space and have period* T *in* t; *and at least one of the two functions* G_1, G_2 *is explicitly a function of* t;

c) *the set of critical points of* $(E_p - 0)$ *is a bounded set*;

d) *the form* $\mathcal{G}(y_1, y_2)$ *is definite and* $\lambda\mathcal{G} < 0$ *where*

$$\lambda = \int_0^{2\pi} \frac{\mathcal{P}(\cos\theta, \sin\theta)}{\mathcal{G}(\cos\theta, \sin\theta)}\, d\theta,$$

or form $\mathcal{G}(y_1, y_2)$ *is not definite and form* $\mathcal{P}(y_1, y_2)$ *is negative (except at the origin) on the straight lines described by*

$$\mathcal{G}(y_1, y_2) = 0.$$

Then for each η *sufficiently close to zero, system* $(E_p - \eta)$ *has at least one solution of period* T.

Theorem 3 is related to but does not overlap the theorems described by Lefschetz [5, Ch. XI], because those theorems describe results for single second order equations which, if transformed into 2-dimensional systems, become

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = H(x_1, x_2, t),$$

where function H is nonlinear. This system is clearly different from system $(E_p - \eta)$.

POLYTECHNIC INSTITUTE OF BROOKLYN, BROOKLYN, N. Y.

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