GENERALIZED SIXPARTITE PROBLEMS

By Jack G. Ceder

In [1] R. C. and E. F. Buck proved that any planar convex body could be divided into six parts of equal area by three concurrent lines. In [2] H. G. Eggleston (and also B. Grünbaum in [3]) stated without proof that this sixpartite problem is true for certain more general measurable sets in the plane. In this article we will show how to extend this result to apply to any set of finite measure* in the plane. More generally we will divide sets into six parts by certain triples of concurrent lines, the parts being commensurable in a specified way with respect to the areas, arc lengths, and angles subtended by these lines.

We will say that a family $\mathfrak{L}$ of lines in the plane admits a *continuous selection* $\mathfrak{M}$ if (1) $\mathfrak{M} \subseteq \mathfrak{L}$; (2) for each direction α (i.e. angle of inclination) there is exactly one member M_α of $\mathfrak{M}$ in that direction; and (3) if $\alpha_n \to \alpha$, then $M_{\alpha_n} \to M_\alpha$ (by the convergence of lines is meant pointwise convergence). A line L is called an *extended diameter* of a convex body C if there is a pair of distinct parallel support lines to C at the points of intersection of L with the boundary of C. A line L is called an *arc-bisector* of a convex body if L divides the boundary into two arcs of equal length. A line L is called an *area-bisector* of a measurable set A if L divides A into two sets of equal measure.

The following theorems, the proofs of which involve similar continuity arguments, provide solutions to some interesting, more general sixpartite problems—in particular, those associated with the above defined kinds of lines.

Theorem 1. *If $\mathfrak{L}$ is any family of lines admitting a continuous selection, and α_1, α_2, and α_3 are any non-negative numbers whose sum is π, then there exist three concurrent lines in $\mathfrak{L}$ subtending angles of α_1, α_2, α_3, α_1, α_2, and α_3 respectively.*

Proof: Let $\mathfrak{M}$ be a continuous selection for $\mathfrak{L}$. For $\alpha \in [0, 2\pi)$, put $f(\alpha) = -\cos\alpha(x - x_0) + \sin\alpha(y - y_0)$, where (x_0, y_0) is any point in M_α and (x, y) is the point of intersection of $M_{\alpha+\alpha_1}$ and $M_{\alpha+\alpha_1+\alpha_2}$. Thus $f(\alpha)$ becomes the familiar directed distance from (x, y) to M_α. From the continuity requirement (3) of a continuous selection it follows that f is continuous. Since the domain of f is connected, the range must be an interval. If $f(0) \neq 0$, then, since $f(0) = -f(\pi)$, $f(0)$ and $f(\pi)$ must have opposite signs. Hence f must have a zero, which yields the three desired concurrent lines.

From this theorem we can then derive the following:

Corollary. If $\mathfrak{L}$ is any one of the following families of lines,

(1) *the extended diameters at a planar convex body*

(2) *the arc-bisectors of a planar convex body*

* By measure, here and in the sequel, we mean any regular measure for which lines have measure zero, in particular Lebesgue measure.

(3) *the arc-bisectors of a bounded set, the closure of the interior of which is connected,*

then the conclusion of Theorem 1 is valid.

Proof: We need only show that each family admits a continuous selection and then apply Theorem 1. It is easily verified that families (2) and (3) are themselves continuous selections. Finally, P. C. Hammer in [4] has shown that one can extract a continuous selection from the extended diameters of a planar convex body.

Next we show

THEOREM 2. *Let C be a planar convex body and let α_1, α_2, and α_3 be any non-negative numbers whose sum is one-half the arc-length of C. Then there exist three concurrent lines (arc-bisectors) which divide the boundary into six arcs of lengths α_1, α_2, α_3, α_1, α_2, and α_3 respectively* (see Fig. 1).

Proof: We can assume that no α_i is 0, otherwise the problem is trivial. Also assume that the origin, Φ, is interior to C. For $z \in BrC$ (the boundary of C), let L_z denote the unique arc-bisector passing through z. Choose z' to be the point on BrC such that the arc length of the arc from z to z' measured positively is equal to α_1. Choose z'' similarly with respect to z' and α_2. Now put $f(z) =$

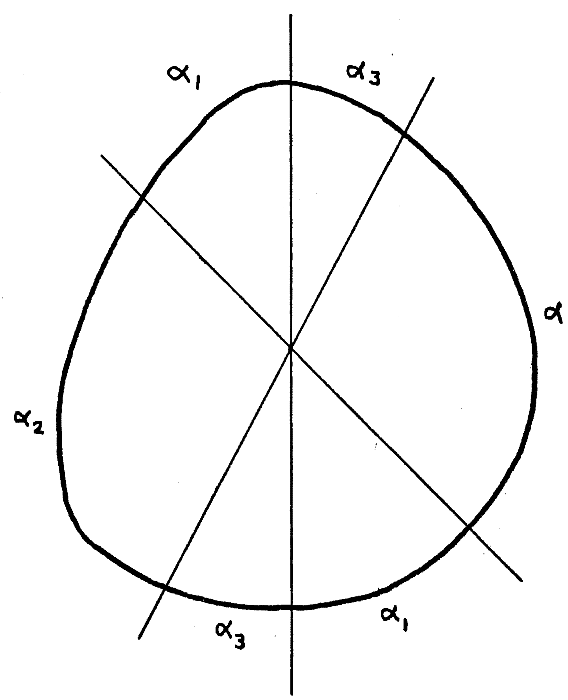

FIG. 1

$-\cos \alpha(x - x_0) + \sin \alpha(y - y_0)$, where (x_0, y_0) is any point on L_z, (x, y) is the point of intersection of $L_{z'}$ and $L_{z''}$, and α is the angle measured positively from the x-axis to the line segment $[\Phi, z]$. The remainder of the proof parallels that of Theorem 1.

Now we have our main theorem, of which the aforementioned result of R. C. and E. F. Buck is a special case. (Fig. 2 illustrates this theorem in the case all a_i are $\frac{1}{6}$ the area.)

THEOREM 3. *Let A be any set of finite measure in the plane, and let a_1, a_2, and a_3 be any non-negative numbers whose sum is one-half the measure of A. Then there exist three concurrent lines (area-bisectors) dividing A into six parts of measures a_1, a_2, a_3, a_1, a_2, and a_3 respectively.*

Proof: Let m be any regular measure in the plane for which lines have measure zero. If $m(A) = 0$, the proof is trivial. So assume $m(A) \neq 0$. We will also assume that all $a_i \neq 0$, as the proof of the opposite case will necessitate a trivial modification of the proof in the case when all $a_i \neq 0$. Analogous to the preceding arguments we might hope that (denoting the union of all area-bisectors in the direction α by W_α) some sort of "directed distance" from the set W_α to the appropriate set $W_{\alpha'} \cap W_{\alpha''}$ would be continuous, and accordingly, application of the continuity reversal argument used in Theorems 1 and 2 would yield the

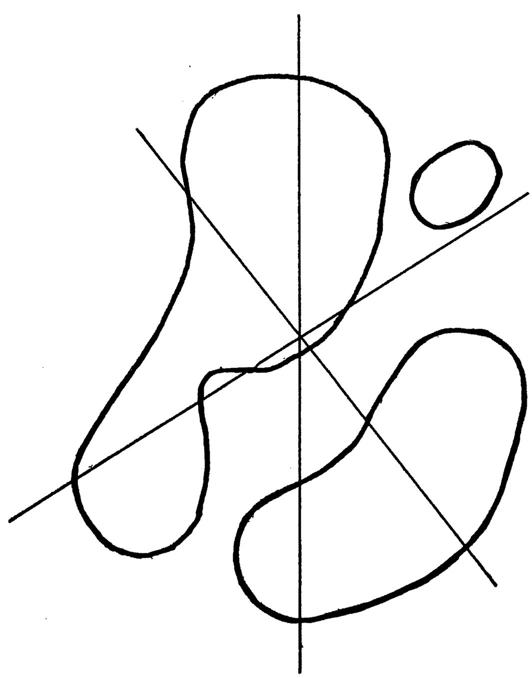

FIG. 2

proof. Unfortunately, however, there may not be enough "continuity" present to permit this, so we must resort to another method of proof, one of "successive approximations." The proof is in three parts.

I. We begin by showing the theorem works when A is a bounded, open, connected set with finite measure. In this case, the set $\mathfrak{L}$ of area-bisectors will obviously be a continuous selection of itself. Denote by L_α the area-bisector of direction α. For each $\alpha \in [0, 2\pi)$, let α' be the least β greater than α (in the counterclockwise sense) for which the two quadrants, swept out by L_α when rotated counterclockwise to L_β, have intersections with A of measure a_1. To show the existence of α', let S_β denote the union of the two quadrants specified above. Clearly $g(\beta) = m(S_\beta \cap A)$ gives a continuous function on $[\alpha, \alpha + \pi)$ for which $g(\alpha) = 0$ and $\lim_{\alpha \to \alpha+\pi} g(\alpha) = m(A)$. Hence there exists a β such that $m(S_\beta \cap A) = 2a_1$. Now suppose $\{\beta_n\}_{n=1}^\infty$ is a decreasing sequence converging to β_0, for which $m(S_{\beta_n} \cap A) = 2a_1$. Then, since $\lim_{n\to\infty} (S_{\beta_n} \cap A) = S_{\beta_0} \cap A$ and $m(A)$ is finite, we have $2a_1 = \lim_{n\to\infty} m(S_{\beta_n} \cap A) = m(S_{\beta_0} \cap A)$. Thus α' exists.

Pick α'' similarly to be the least β greater than α' for which both of the quadrants formed by rotating $L_{\alpha'}$ counterclockwise to L_β have intersection with A of measure a_2. Now, let $f(\alpha)$, as in the previous theorems, be the directed distance from L_α to the point of intersection of $L_{\alpha'}$ and $L_{\alpha''}$. The rest of the proof is similar to that of Theorem 1.

II. Assuming now the result of part I, we will show that the theorem is valid for any bounded set of finite measure by suitably approximating such sets by bounded, open, connected, measurable sets and then employing a limit argument. For each n we choose G_n to be an open, connected set containing A, so that $m(G_n - A) < 1/n$. To do this (by the regularity of m) pick 0_n to be an open set containing A so that $m(0_n - A) < 1/2n$. Since 0_n is the union of some sequence of open discs $\{S_i\}_{i=1}^\infty$, pick C_i to be an open strip joining S_i to S_{i+1}, so that $m(C_i) < 1/n2^{i+1}$. Next put $G_n = 0_n \cup (\bigcup_{i=1}^\infty C_i)$ to obtain the desired open, connected sets.

By part I, each G_n can be divided by three concurrent lines making areas of $a_1 + \delta_{n/6}$, $a_2 + \delta_{n/6}$, $a_3 + \delta_{n/6}$, etc., where $\delta_n = m(G_n) - m(A)$. For each n let L_1^n denote one of the three lines which is on the clockwise side of a sector of area $a_1 + \delta_{n/6}$. Let $\alpha_1^n \in [0, \pi)$ be the direction of L_1^n. Let L_2^n be the next line in the positive sense, and put $\alpha_2^n = \alpha_1^n + \alpha$, where α is the angle between L_1^n and L_2^n. Likewise choose L_3^n and α_3^n. Finally let z_n be the point of intersection of L_1^n, L_2^n and L_3^n.

The sequence $\{\alpha_1^n\}_{n=1}^\infty$ in the "circle" $[0, 2\pi)$ has a convergence subsequence (which we may assume, for expediency, to be the original sequence) converging to some $\alpha_0 \in [0, 2\pi)$. Without loss of generality we can assume that all α_1^n are different from α_0 and $\alpha_0 + \pi$ (for, if $\alpha_1^n = \alpha_0$ or $\alpha_0 + \pi$ for infinitely many n, we easily get a subsequence of $\{L_1^n\}_{n=1}^\infty$ which converges). So let w_n be the point of intersection of L_1^n and L_1^1. And pick $z \in L_1^1$ so that $z \neq w_n$ for any n. Now, clearly, if $\{w_n\}_{n=1}^\infty$ is bounded, we will obtain a subsequence of $\{L_1^n\}_{n=1}^\infty$ which converges to some line L_1^0 of direction α_0. So assume $\{w_n\}_{n=1}^\infty$ is unbounded.

Without loss of generality we can assume $w_n \to \infty$. Let H_n be the open half-plane determined by L_1^n which does not contain z. Since A has finite measure, we have

$$0 < \tfrac{1}{2}\, m(A) \leq \limsup_{n\to\infty} m(H_n \cap A) \leq m(\limsup_{n\to\infty} (H_n \cap A)).$$

It follows then there must be a point y in $\limsup_{n\to\infty} (H_n \cap A)$ which is not in L_1^1. But this implies that y and z are on opposite sides of infinitely many members of $\{L_1^n\}_{n=1}^\infty$. But since $w_n \to \infty$, this means that a subsequence of $\{L_1^n\}_{n=1}^\infty$ converges to L_1^1, which contradicts the fact that $\alpha_1^1 \neq \alpha_0$ and $\alpha_0 + \pi$. Hence, $\{L_1^n\}_{n=1}^\infty$ has a convergent subsequence.

Repeating similar arguments on the other two sequences of lines, we eventually obtain a subsequence $\{j_n\}_{n=1}^\infty$, lines L_1^0, L_2^0, L_3^0, and a $z_0 \in L_1^0 \cap L_2^0 \cap L_3^0$ such that $L_i^{j_n} \to L_i^0$ for each i and $z_{j_n} \to z_0$. To show that L_1^0, L_2^0 and L_3^0 give the desired division of A, let S_n denote the union of the two quadrants "between" L_1^n and L_2^n. Then it is easily verified that $S_0 \cap A = \limsup_{n\to\infty} (S_n \cap A) = \liminf_{n\to\infty} (S_n \cap A)$. Hence, $m(S_0 \cap A) = \lim_{n\to\infty} m(S_n \cap A) = 2 \lim_{n\to\infty} (a_1 + \delta_n/6) = 2a_1$, which shows that L_1^0 and L_2^1 are distinct and that the sector "between" them has the desired measure. Likewise we handle the other pairs of lines, which completes the proof of part II.

III. Finally we show that the theorem is valid for arbitrary finitely measurable sets. For each n, find, by the regularity of m, a compact set A_n contained in A for which $m(A - A_n) < 1/n$. Put $\delta_n = m(A) - m(A_n)$. By part II divide A_n into sectors of measures $a_1 - \delta_n/6$, $a_2 - \delta_n/6$, $a_3 - \delta_n/6$, etc. Now, repeating the limit argument in part II, we obtain the desired division of A, which completes the proof.

Theorem 3 may be false for sets of infinite measure, e.g., Lebesgue measure and the set $\left\{(x, y) : 0 \leq y \leq \frac{1}{x}, x \geq 1\right\}$. Moreover, neither of the three above theorems can be extended to the case of four numbers and four concurrent lines, a triangle providing the counter example in each case.

The foregoing suggests many other interesting unsolved sixpartite problems. For example, given six positive numbers whose sum is the area (or arc length) of a finitely measurable set (or convex body), are there always three concurrent lines dividing the set into six parts having area (or arc lengths) equal to the given six numbers?

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BIBLIOGRAPHY

[1] R. C. and E. F. BUCK, *Equipartition of convex sets*, Math. Mag., **22** (1949), 195–98.
[2] H. G. EGGLESTON, *Some properties of triangles as extremal convex curves*, J. London Math. Soc., **28** (1953), 32–36.
[3] B. GRÜNBAUM, *Measure of symmetry for convex sets*, appearing in Convexity, (vol. 7 of Proc. of Symposia in Pure Math. A.M.S. 1963), p. 233–70.
[4] P. C. HAMMER, *Convex bodies associated with a convex body*, Proc. A.M.S., **2** (1951), 781–95.