A NOTE ON QUASI-LOCAL RINGS

By Rolando E. Peinado and R. B. Killgrove

A commutative ring A with more than one element is called a *quasi-local* ring if it contains an identity element and the non-units of A form an ideal. It can be shown that this ideal is the only maximal ideal of A. Since the Jacobson radical, $J(A)$ of A is the intersection of the maximal ideals of A, in a quasi-local ring, $J(A)$ consists of the non-units of A. A Noetherian quasi-local ring is called a *local* ring. By a well known theorem of Krull, in a local ring the intersection of the powers of the Jacobson radical is zero; i.e., $\bigcap_m [J(A)]^m = 0$ (see [3, Cor. 2, p. 217]). The following example will show that the result for local rings quoted above does not always hold for quasi-local rings. This example is constructed in the manner of [1] and [2].*

Consider the set of symbols $\{x_{r_i,s_i}\}$, $i = 1, 2, \cdots$, where r_i/s_i is a rational number in the open interval $(0, 1)$ and the greatest common divisor (g.c.d.) of r_i and s_i is one. A finite sequence w of symbols x_{r_i,s_i} is called a *word*. By the length of a word, written $\ell(w)$, we mean the number of symbols x_{r_i,s_i} in w. Two words w and v are the *same* if $\ell(w) = \ell(v)$ and corresponding places in w and v are occupied by the same symbols. Let Z_2 represent the ring of rational integers *modulo* two.

Let R be the set of all finite sums of words w, built from the symbols x_{r_i,s_i} over Z_2. That is, R is the set of all "polynomials" in $\{x_{r_i,s_i}\}$ with coefficients in Z_2. Note that R is assumed to contain words of zero length, that is, the empty word, consisting of no symbols. Hence 1 and 0 are elements of R. Let

$$\alpha = \textstyle\sum_{i=1}^{n} a_i w_i \quad \text{and} \quad \beta = \sum_{j=1}^{m} b_j v_j \tag{1}$$

(where $a_i, b_j \in Z_2$ and w_i, v_j are words) be arbitrary elements of R. A relation $\equiv$, called *equality*, in R is defined as follows: $\alpha \equiv \beta$ if and only if, for each word w in α, there exists a word v in β, such that v is the same word as w, and the sum of the coefficients of w in α is equal to the sum of the coefficients of v in β; or if there is not a word in β that is the same as w in α, the sum of the coefficients of w in α is zero. (Recall that the coefficients of words in R are added modulo two.) It is easy to show that the relation $\equiv$ among members of R is an equivalence relation on R. Define the following two binary operations in R. For α and β, members of R as in (1), define the *sum* of α and β by

$$\alpha + \beta = \textstyle\sum_{i=1}^{n} a_i w_i + \sum_{j=1}^{m} b_j v_j = \sum_{i=1}^{n+m} a_i w_i \tag{2}$$

where $a_{n+k} w_{n+k} = b_k v_k$ for $k = 1, \cdots, m$. Define the *product* of α and β by

$$\alpha\beta = (\textstyle\sum_{i=1}^{n} a_i w_i)(\sum_{j=1}^{m} b_j v_j) = \sum_{i,j=1}^{n,m} a_i b_j w_i v_j \tag{3}$$

* This example is constructed without use of valuation rings. It is clear that if R_v is a non-discrete valuation ring, then R_v is a quasi-local ring, and if M_v is the ideal formed by the non-units of R_v, then $\bigcap_n M_v^n \neq 0$.

where $\mathscr{w}_i\mathscr{v}_j$ is the word obtained by writing the sequence $\mathscr{v}_j$ after the sequence $\mathscr{w}_i$ of symbols x_{r_i, s_i}.

Let $[\alpha]$ be the equivalence class of elements equal to α, determined by the equivalence relation $\equiv$ defined above. Define, as is customary, $[\alpha] \oplus [\beta] = [\alpha + \beta]$ and $[\alpha] \odot [\beta] = [\alpha\beta]$. Then the set $R*$ of equivalence classes, together with the operations $\oplus$ and $\odot$ defined above, forms a non-commutative ring with identity. (This fact is easily verified.) Now, for the sake of simplicity, let us not distinguish between a class of equivalent elements of R and any member of the class. Henceforth we will talk about the ring $\langle R, +, \cdot\rangle$ when we really mean the ring $\langle R*, \oplus, \odot\rangle$.

Let p_i/q_i and r_j/s_j be rational numbers on the open interval $(0, 1)$ such that the g.c.d. $(p_i, q_i) =$ g.c.d. $(r_j, s_j) = 1$. Define

$$u = p_is_j + q_ir_j/\text{g.c.d. } (p_is_j + q_ir_j, q_is_j), \quad \text{and} \tag{4}$$
$$v = q_is_j/\text{g.c.d. } (p_is_j + q_ir_j, q_is_j).$$

An easy calculation will show that u and v are rational integers and g.c.d. $(u, v) = 1$. Let T be the set of words of R of the form $\{x_{p_i, q_i}x_{r_j, s_j}\}$. A word $\mathscr{w}$ in R is said to be in *normal form*, written $\mathbf{N}(\mathscr{w})$, if and only if it does not contain members from T. An element α of R is said to be in *normal form*, $\mathbf{N}(\alpha)$, if and only if each word $\mathscr{w}$ of α is in normal form. For any word $\mathscr{w}$, let $\mathscr{w}_1$ be the word obtained from $\mathscr{w}$ by eliminating a member of T from $\mathscr{w}$, using one of the relations

$$x_{p_i, q_i}x_{r_j, s_j} - x_{u,v} = 0, \quad \text{if } u < v, \quad \text{and} \quad x_{p_i, q_i}x_{r_j, s_j} = 0, \quad \text{if } u \geqslant v, \tag{5}$$

u and v being as defined in (4). Call this process an *elementary transformation*, written $\mathscr{w} \rightarrow \mathscr{w}_1$. Two words $\mathscr{w}$ and $\mathscr{v}$ are said to be *similar* if there exists a finite sequence of words $\mathscr{w}, \mathscr{w}_1, \mathscr{w}_2, \cdots, \mathscr{w}_n, \mathscr{v}$ in R such that each member of the sequence is obtained from the preceding one by an elementary transformation. Two elements α and β of R are said to be *similar* if, for every word $\mathscr{w}$ of α, there is a word $\mathscr{v}$ in β similar to $\mathscr{w}$.

Let H be the subset of R, consisting of all finite sums and products of left members of (5). H will be a two-sided ideal of R if it can be shown that a solution exists for the decision problem as to whether or not an element α of R belongs to H. Let α be similar to β, and α and β elements of R. If a unique normal form $\mathbf{N}(\alpha)$ exists, then $\mathbf{N}(\alpha) = \mathbf{N}(\beta)$. Clearly $\alpha \in H$ if and only if $\mathbf{N}(\alpha) = 0$. Hence the existence of such a unique normal form gives us the solution for the above decision problem. As a consequence of the definitions of addition and multiplication in R as well as the definition of normal form, we have for α, β in R (if $\mathbf{N}(\alpha)$ and $\mathbf{N}(\beta)$ exist and are unique)

$$\mathbf{N}(\alpha + \beta) = \mathbf{N}(\alpha) + \mathbf{N}(\beta) \quad \text{and} \quad \mathbf{N}(\alpha\beta) = \mathbf{N}[\mathbf{N}(\alpha)\cdot\mathbf{N}(\beta)].$$

Therefore, to show the existence of such unique normal form $\mathbf{N}(\alpha)$ for each α in R, it is sufficient to consider the words $\mathscr{w}$ of α.

Let $\mathscr{w}$ be a word in R; recall that the number of symbols x_{r_i,s_i} which appear in $\mathscr{w}$ is called the length of $\mathscr{w}$, written $\ell(\mathscr{w})$. If $\mathscr{w}$ is the empty word, $\ell(\mathscr{w}) = 0$. We will prove the existence and uniqueness of $\mathbf{N}(\mathscr{w})$ by induction on the length of $\mathscr{w}$.

It is clear that if $\ell(\mathscr{w}) \leqslant 1$, then $\mathscr{w} = \mathbf{N}(\mathscr{w})$; that is, $\mathscr{w}$ is already in normal form and is unique. Assume that, for all words $\mathscr{w}$ such that $\ell(\mathscr{w}) \leqslant n - 1$, $\mathbf{N}(\mathscr{w})$ exists and is unique. Let $\mathscr{v}$ be a word of R such that $\ell(\mathscr{v}) = n$, and let

$$\mathscr{v} = \mathscr{w}x_{p_i,q_i}x_{r_j,s_j}$$

where $\ell(\mathscr{w}) \leqslant n - 1$. Hence, by the induction hypothesis, $\mathbf{N}(\mathscr{w})$ exists and is unique. We have to consider two cases, for u and v defined as in (4).

Case 1. If $u \geqslant v$, then

$$\mathscr{w}x_{p_i,q_i}x_{r_j,s_j} \rightarrow \mathscr{w}\cdot 0 = 0$$

Hence $\mathscr{v} \rightarrow \mathscr{w}\cdot 0 = 0$. But $\mathbf{N}(0) = 0$ and, since $\mathscr{v}$ is similar to 0, $\mathbf{N}(\mathscr{v}) = \mathbf{N}(0) = 0$. Hence $\mathbf{N}(\mathscr{v})$ exists and is unique.

Case 2. If $u < v$, then

$$\mathscr{w}x_{p_i,q_i}x_{r_j,s_j} \rightarrow \mathscr{w}x_{u,v}$$

and $\ell(\mathscr{w}x_{u,v}) \leqslant n - 1$, and thus, by the induction hypothesis, $\mathbf{N}(\mathscr{w}x_{u,v})$ exists and is unique. Now, since $\mathscr{v}$ and $\mathscr{w}x_{u,v}$ are similar, $\mathbf{N}(\mathscr{v}) = \mathbf{N}(\mathscr{w}x_{u,v})$. Hence $\mathbf{N}(\mathscr{v})$ exists and is unique.

The above arguments show that H is a (two-sided) ideal of R. Therefore $R/H = A$ is well defined. As can be observed, the elements of A are of the form $\sum_{i=1}^{n} a_i\mathscr{w}_i$, where $\mathscr{w}_i = x_{r_i,s_i}$ or $\mathscr{w}_i$ is the empty word, $a_i \in Z_2$. A is a commutative ring with identity, namely $1 \in A$. The units of A are of the form $1 + \sum_{i=1}^{n} a_i\mathscr{w}_i$, $a_i \in Z_2$, $\mathscr{w}_i = x_{r_i,s_i}$ or $\mathscr{w}_i$ is the empty word. Clearly the non-units of A are of the form $\sum_{i=1}^{n} a_i x_{r_i,s_i}$, $a_i \in Z_2$, and they form an ideal in A, which, as remarked before, is equal to $J(A)$. Hence A is a quasi-local ring. Now, since

$$x_{1,2} = x_{1,2m}x_{m-1,2m},$$

$x_{1,2}$ belongs to $[J(A)]^m$ for any positive rational integer m. Therefore

$$\bigcap_m [J(A)]^m \neq 0.$$

UNIVERSITY OF IOWA, IOWA CITY
SAN DIEGO STATE COLLEGE, SAN DIEGO, CALIF.

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