THE SPACE OF CONTINUOUS SEMINORMS

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1. Introduction

A normed linear space together with the topology induced by the norm α is denoted by (X, α). Let $N(X)$ be the set of all continuous seminorms defined in (X, α), and let $E(\alpha)$ be the set of all norms equivalent to α. We metrize $N(X)$ by $d_\alpha(p, q) = \sup_{\alpha(x) \leqq 1} |p(x) - q(x)|$. In the dual space (X^*, α^*) one defines similarly $N(X^*)$ and $E(\alpha^*)$. (α^* is the conjugate norm of α). In this paper we consider the map $\alpha \to \alpha^*$ from $E(\alpha)$ into $E(\alpha^*)$ which is shown to be continuous (Prop 2.4). We also show that $E(\alpha)$ is an open dense subset of $N(X)$ (Prop 2.3). It follows from this result that if (X, α) is a separable Banach space, then the set of smooth norms equivalent to α is dense in $N(X)$. It has been shown by V. Klee in [5] and M. Day in [2] that if (X, α) is separable, then there is at least one norm equivalent to α which is both smooth and rotund and has a rotund conjugate norm. We recall that a norm β is smooth if there is a unique tangent hyperplane at each point in the unit sphere; and that a norm β is rotund if the unit sphere contains no line segments. Throughout this paper function always means continuous function.

2. The space $N(X)$

Let (X, α) be a normed linear space, and let $P(X) = \{f : X \to R \mid f(tx) = tf(x)$ for $t \geqq 0\}$. It is clear that each $f \in P(X)$ is bounded in the unit ball $B_\alpha = \{x \mid \alpha(x) < 1\}$. Thus $\| f \|_\alpha = \sup_{\alpha(x) \leqq 1} |f(x)| = \sup_{\alpha(x)=1} |f(x)|$ defines a norm in $P(X)$, so $P(X)$ becomes a Banach space. If α is equivalent to β then $\| \ \|_\alpha$ is equivalent to $\| \ \|_\beta$. We can consider $N(X)$ as a subspace of $P(X)$ with the metric $d_\alpha(p, q) = \| p - q \|_\alpha$.

Proposition 2.1. *Let (X, α) be a normed linear space. Then a continuous norm β is equivalent to α if and only if there is a constant $M > 0$ such that $\alpha(x) \leqq M\beta(x)$ for all $x \in X$.*

Proposition 2.2. *Let (X, α) be a normed linear space, and let $N(X)$ be the space of all continuous seminorms. Then,*

(i) *the operations $(p, q) \to p + q$ and $(t, q) \to tq$ are defined in $N(X)$ and are continuous;*

(ii) *$N(X)$ is a closed convex cone in $P(X)$; and*

(iii) *if β is equivalent to α, then $\beta + N(X) \subset E(\alpha)(E(\alpha)$ is the set of norms equivalent to α).*

Proof: (i) and (ii) are trivial, and (iii) follows from Proposition 2.1.

Proposition 2.3. *Let (X, α) be a normed linear space, and let $E(\alpha)$ be the set of all norms equivalent to α. Then $E(\alpha)$ is an open, dense, connected subset of $N(X)$.*

Proof: Let $h:N(X) \to (0, \infty)$ be defined by $h(p) = \inf_{\alpha(x)=1} p(x)$.

(a) $h(p) > 0$ if and only if $p \in E(\alpha)$. If $p \in E(\alpha)$ then $\alpha(x) \leqq Mp(x)$, so $p(x) \geqq 1/M > 0$ for all x such that $\alpha(x) = 1$. Conversely, if $h(p) > 0$ then $p(x) \geqq M > 0$ for all x such that $\alpha(x) = 1$, so for any y one has $p(y/\alpha(y)) \geqq M > 0$ and $\alpha(y) \leqq 1/Mp(y)$; i.e., $p \in E(\alpha)$.

(b) h is continuous. Let $p_0 \in N(X)$ and let $\epsilon > 0$ and let $p \in N(X)$ be such that $d_\alpha(p, p_0) < \epsilon/2$. Then $|p(x) - p_0(x)| < \epsilon/2$ for all x such that $\alpha(x) = 1$. From $p(x) < (\epsilon/2) + p_0(x)$ one obtains $\inf_{\alpha(x)=1} p(x) < (\epsilon/2) + p_0(y)$ for every fixed y such that $\alpha(y) = 1$. Find some y_1 with $\alpha(y_1) = 1$ such that $\inf_{\alpha(y)=1} p_0(y) + (\epsilon/2) \geqq p_0(y_1)$. For such y_1 one has $\inf_{\alpha(x)=1} p(x) \leqq (\epsilon/2) + p_0(y_1) < \epsilon + \inf_{\alpha(x)=1} p_0(x)$; i.e., $h(p) - h(p_0) < \epsilon$. From the inequality $p_0(x) < (\epsilon/2) + p(x)$ for all x with $\alpha(x) = 1$ one obtains, in a similar way, the inequality $h(p_0) - h(p) < \epsilon$. Thus we have proved that $|h(p) - h(p_0)| < \epsilon$ whenever $d_\alpha(p, p_0) < \epsilon/2$, i.e., that h is continuous.

(c) $E(\alpha)$ is open since $E(\alpha) = h^{-1}(0, \infty)$.

(d) $E(\alpha)$ is dense in $N(X)$. For any $p \in N(X)$, one has $p_t = t\alpha + (1 - t)p \in E(\alpha)$, if $0 < t \leqq 1$. Thus $p_t - p = t(\alpha - p)$, and $\sup_{\alpha(x)=1} |p_t(x) - p(x)| = t \sup_{\alpha(x)=1} |\alpha(x) - p(x)| \to 0$ as $t \to 0$; so $E(\alpha)$ is dense in $N(X)$.

We could also consider the space $N(X^*)$ of all continuous seminorms in (X^*, α^*). It is easy to see that if β is equivalent to α then β^* is equivalent to α^*.

PROPOSITION 2.4. *Let $h:E(\alpha) \to E(\alpha^*)$ be the map which assigns to each $\beta \in E(\alpha)$ its conjugate norm β^*. Then h is a homeomorphism from $E(\alpha)$ into $E(\alpha^*)$. If X is reflexive, then h is a homeomorphism onto.*

Proof: Let $\beta_0 \in E(\alpha)$, let $\epsilon > 0$, and let $d = d_{\beta_0}$ be the metric induced by β_0 in $N(X)$. Let $\delta = \epsilon/(1 + \epsilon)$. For each β such that $d(\beta, \beta_0) < \delta$, and for each x with $\beta_0(x) = 1$, one has $|\beta(x) - \beta_0(x)| = |\beta(x) - 1| < \delta$. Thus $1 - \delta < \beta(x) < 1 + \delta$, whenever $\beta_0(x) = 1$. If $\beta(x) = 1$ and $d(\beta, \beta_0) < \delta$, one has $1 - \delta < \beta(x/\beta_0(x)) < 1 + \delta$. Therefore, $\beta(x) = 1$ and $d(\beta, \beta_0) < \delta$ implies $1/(1 + \delta) < \beta_0(x) < 1/(1 - \delta)$; i.e., $1 - \epsilon < \beta_0(x) < 1 + \epsilon$. If $d(\beta, \beta_0) < \delta$ and ${\beta_0}^*(f) = 1$ one has $|{\beta_0}^*(f) - \beta^*(f)| = |1 - \sup_{\beta(x)=1} f \cdot x|$. On the other hand $S_\beta = \{x \mid \beta(x) = 1\}$ is contained in the annulus $\{y \mid 1 - \epsilon < \beta_0(y) < 1 + \epsilon\}$; so $1 - \epsilon \leqq \sup_{\beta(x)=1} |f \cdot x| \leqq 1 + \epsilon$. Therefore, $|{\beta_0}^*(f) - \beta^*(f)| \leqq \epsilon$ and $d({\beta_0}^*, \beta^*) \leqq \epsilon$, whenever $d(\beta, \beta_0) < \delta = \epsilon/(1 + \epsilon)$. This shows that $\beta \to \beta^*$ is continuous.

A similar map from $E(\alpha^*)$ into $E(\alpha^{**})$ can be defined. If we identify X with $j(X)$ (j is the canonical imbedding of X into X^{**}) then the restriction of β^{**} to X is precisely β. Therefore h has a continuous inverse. If X is reflexive, then $X = X^{**}$ and h is onto.

3. Smoothness and rotundity

We follow here the terminology of Day ([2]). A norm β is smooth if there is a unique supporting hyperplane at each point x of the unit sphere $S_\beta =$

$\{y \mid \beta(y) = 1\}$. And a norm β is rotund if the unit sphere contains no line segments. The following proposition can be found in [2].

PROPOSITION 3.1. (i) *A norm β is smooth if and only if β is differentiable in any planar section through the origin.*

(ii) *A norm β is rotund if and only if for any x_1, $x_2 \in S_\beta$ one has $\beta((x_1 + x_2)/2) < 1$, where $x_1 \neq x_2$.*

(iii) *A norm β is rotund if and only if $\beta(x_1 + x_2) < \beta(x_1) + \beta(x_2)$ whenever $x_1 \neq x_2$.*

PROPOSITION 3.2. *Let (X, α) be a normed linear space, let $N(X)$ be the space of all continuous seminorms, and let R be the set of all rotund norms in X. Assume R is non-empty. Then,*

(i) *$R + N \subset R$ and $tR \subset R$ for all $t > 0$;*

(ii) *R is a dense, convex subset of $N(X)$;*

(iii) *the set $R \cap E(\alpha)$ of all rotund norms equivalent to α is convex and dense in $N(X)$.*

Proof: (i) follows from Proposition 3.1. It is clear that R is convex. If p is any element of $N(X)$ and β_0 is rotund, then $p_t = t\beta_0 + (1 - t)p$ is rotund for all $0 < t < 1$; and for any $x \in S_\alpha$ one has $|p_t(x) - p(x)| = t\,|\beta_0(x) - p(x)|$. Therefore $d_\alpha(p_t, p) = t\,d(\beta_0, p) \to 0$ as $t \to 0$. Part (iii) now follows from Proposition 2.3.

PROPOSITION 3.3 (see V. L. Klee [4]). *Let (X, α) be a normed linear space and let β be any norm equivalent to α. Then*

(i) *if β^* is rotund, then β is smooth;*

(ii) *if β^* is smooth, then β is rotund.*

PROPOSITION 3.4. *Let (X, α) be a normed linear space. Assume there is at least one rotund conjugate norm. Then the set of all smooth norms in $E(\alpha)$ having a rotund conjugate is dense in $N(X)$ (the space of continuous seminorms).*

Proof: Let β_0^* be a fixed rotund conjugate norm and let β^* be any conjugate norm. Since every conjugate norm is lower semicontinuous in the weak topology (see [2]), it follows that $t\beta_0^* + (1 - t)\beta^* = \gamma_t^*$ is also a conjugate norm for all $0 \leqq t \leqq 1$. Moreover, γ_t^* is rotund for all $0 < t \leqq 1$ (Proposition 3.2). The same argument used in the proof of part (ii) of Proposition 3.2 shows that the set of all rotund conjugate norms is dense in the set of all conjugate norms. The proposition now follows from Propositions 2.4 and 3.2.

It has been shown by M. Day in [2] that if (X, α) is separable then there exists a rotund conjugate norm. This together with Proposition 3.4 yields

PROPOSITION 3.5. *Assume (X, α) is separable. Then the set of smooth norms in $E(\alpha)$ having a rotund conjugate is dense in $N(X)$ (the space of continuous seminorms).*

Remark. A norm β is said to be Gatteaux differentiable if for each $x \neq 0$ there is some $f(x) \in X^*$ such that for any $u \in X \lim_{t \to 0} (\beta(x + \lambda u) - \beta(x)/t) = f(x)u$. A norm β is said to be Frechet differentiable if, for each $x \neq 0$, there is some $f(x) \in X^*$ such that for any $u \in X \, |(\beta(x + u) - \beta(x) - f(x) \cdot u)/\beta(u)| \to 0$ as $\beta(u) \to 0$. It is known that β is smooth if and only if β is Gatteaux differentiable. On the other hand, if β is Frechet differentiable, then β is Gatteaux differentiable. The author has shown (Bull. Amer. Math. Soc. **70**(1964)) that no norm equivalent to the usual one in $C[0, 1]$ is Frechet differentiable. On the other hand, Proposition 3.5 shows that the set of Gatteaux differentiable norms in $C[0, 1]$ is dense in $N(X)$.

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REFERENCES

[1] D. F. CUDIA, *Rotundity*, Proc. Symp. Pure Math., 7 (1963), 73–97.

[2] M. M. DAY, *Strict convexity and smoothness of normed linear spaces*, Trans. Amer. Math. Society, **78** (1955).

[3] ———, Normed linear spaces, Academic Press Inc., New York, 1962.

[4] V. L. KLEE, *Convex bodies and periodic homeomorphisms in Hilbert spaces*, Trans. Amer. Math. Soc., **74** (1953), 16–43.

[5] ———, *Some new results on smoothness and rotundity in normed linear spaces*, Math. Ann., **139** (1959), 51–63.